DIPLOMARBEIT

Long-Time Asymptotics for the KdV Equation

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Chapter 0

Introduction

One of the most famous examples of nonlinear wave equations is the Korteweg–de Vries (KdV) equation

$$q_t(x,t) = 6q(x,t)q_x(x,t) - q_{xxx}(x,t). \tag{0.1}$$

$(x,t) \in \mathbb{R} \times \mathbb{R}$. Here the subscripts denote the differentiation with respect to the corresponding variables. We will consider real-valued solutions $q(x,t)$ corresponding to rapidly decaying initial conditions, for example,

$$q(x,0) \in \mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) | \sup_x |x^\alpha (\partial_\beta f)(x)| < \infty, \alpha, \beta \in \mathbb{N}_0\}. \tag{0.2}$$

The KdV equation occurs in the context of models for small amplitudes, long (water) waves, collision-free hydromagnetic waves and ion-acoustic waves in plasmas.

It is well-known that the KdV equation can be solved by the inverse scattering method applied to the associated Schrödinger operator

$$H(t) = -\frac{d^2}{dx^2} + q(x,t). \tag{0.3}$$

The scattering data (see e.g. [7], [15]) consists of the reflection coefficient $R(k,t)$, a finite number of eigenvalues $-\kappa_j^2$ with $0 < \kappa_1 < \kappa_2 < \cdots < \kappa_N$ and norming constants $\gamma_j(t)$.

The aim of my diploma thesis is to compute the long-time asymptotics for the KdV equation in the soliton and the similarity region. Our approach is based on the nonlinear steepest descent method for oscillatory Riemann–Hilbert problems from Deift and Zhou [9]. We closely follow the recent review article [13], where Krüger and Teschl applied this method to compute the long-time asymptotics for the Toda lattice. One of the many differences here is the fact that the jump contour of the associated Riemann–Hilbert problem is no longer bounded.

For computing the long-time behavior of the KdV equation there are four cases to distinguish.

(i) Soliton Region:
For $x/t > C$ for some $C > 0$,

$$q(x,t) \sim -2\sum_{j=1}^N \frac{\kappa_j^2}{\cosh^2(\kappa_j x - 4\kappa_j^3 t - p_j)}, \tag{0.4}$$

where the phase shifts $p_j = \frac{1}{2}\log[\frac{\gamma_j^2}{2\kappa_j}\prod_{l=j+1}^{N}\left(\frac{\kappa_l-\kappa_j}{\kappa_l+\kappa_j}\right)^2]$ and $\gamma_j = \gamma_j(0)$

This result, which will be proven in Theorem 3.5, has an interesting physical interpretation:

In general every term in our asymptotic formula describes a wave. Considering only two of them, with the smaller one to the right, after a certain time the waves will overlap (i.e. the bigger one catches up) and next the bigger one will separate from the smaller one. Then gradually the waves regain their initial speed and shape. The only permanent effect of this interaction is that the bigger one is shifted to the right and the smaller one to the left.

A proof using inverse scattering based on the Gel'fand-Levitan-Marchenko equation can be found in Eckhaus and Schuur [11] (see also [18]).

(ii) Self-Similar Region:
$\left|x/(3t)^{1/3}\right| \le C$ for some $C > 0$
The solution in this region is connected with the Painléve II transcendent. The investigation of this connection can be found in Segur and Ablowitz [19].

(iii) Collisionless Shock Region:
$x < 0$ and for $C^{-1} < \frac{-x}{(3t)^{1/3}(\log(t))^{2/3}} < C$, for some $C > 1$
This region only occurs in the generic case (i.e., when $R(0) = -1$) and the long-time asymptotic is investigated in Deift, Venakides and Zhou [8].

(iv) Similarity Region:
For $x/t < -C$ for some $C > 0$,

$$q(x,t) \sim \left(\frac{4\nu(k_0)k_0}{3t}\right)^{1/2} \sin(16tk_0^3 - \nu(k_0)\log(192tk_0^3) + \delta(k_0)), \quad (0.5)$$

with

$$\begin{aligned}\nu(k_0) =& -\frac{1}{2\pi}\log(1-|R(k_0)|^2),\\ \delta(k_0) =& \frac{\pi}{4} - \arg(R(k_0)) + \arg(\Gamma(\mathrm{i}\nu(k_0))) + 4\sum_{j=1}^{N}\arctan\left(\frac{\kappa_j}{k_0}\right)\\ &+ \frac{1}{\pi}\int_{-k_0}^{k_0}\log(|\zeta-k_0|)d\log(1-|R(\zeta)|^2).\end{aligned}$$

Here $k_0 = \sqrt{-\frac{x}{12t}}$ denotes the stationary phase point, $R(k) = R(k, t = 0)$ the reflection coefficient, and Γ the Gamma function.
This will be proven in Theorem 4.3.
An analytic discussion can be found in Ablowitz and Segur [1] and in Buslaev and Sukhanov [4].

Note that if $q(x,t)$ solves the KdV equation, then so does $q(-x,-t)$. Therefore it suffices to investigate the case $t \to \infty$.

The content of this thesis is as follows:

In Chapter 1 we derive the Riemann–Hilbert problem from scattering theory, where the eigenvalues are added by appropriate pole conditions, which are then turned into jumps.

Chapter 2 proves a uniqueness result for symmetric Riemann–Hilbert problems. In general, there is a well-known non-uniqueness issue for the involved Riemann–Hilbert problems (see e.g. [3, Chap. 38]).

Chapter 3 demonstrates how to conjugate our Riemann–Hilbert problem and deform our contour such that the jumps are exponentially decreasing away from the stationary phase points. Moreover, the asymptotics in the soliton region are computed.

In Chapter 4 we compute the asymptotics in the similarity region. The crucial step is to reduce the given Riemann–Hilbert problem to one or more Riemann–Hilbert problems localized at the stationary phase points. These local Riemann–Hilbert problems can be analyzed and controlled individually.

In Chapter 5 we consider the case where the reflection coefficient has no analytic extension to a neighborhood of the real axis and so we show how to split it in an analytic part plus a small rest.

Appendix A investigates the solution on a small cross, which occurs in the neighborhood of the stationary phase points.

In Appendix B we have a close look at the connection between singular integral equations and Riemann–Hilbert problems, which is needed for computing the asymptotics.

Thanks

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Errata

This version differs a bit from the one, I submitted as my diploma thesis, since I have corrected some small mistakes.

Chapter 1

The Inverse scattering transform and the Riemann–Hilbert problem

In this chapter we want to derive the Riemann–Hilbert problem from scattering theory. The eigenvalues will be added by appropriate pole conditions which are then turned into jumps following Deift, Kamvissis, Kriecherbauer, and Zhou [6] in the case of the Toda lattice (see also Krüger and Teschl [14]).

1.1 Results from scattering theory

For the necessary results from scattering theory respectively the inverse scattering transform for the KdV equation we refer to [7] and [15].

We consider real-valued solutions $q(x,t)$ of the KdV equation (0.1), which decay rapidly, that is

$$\max_{|t|\le T}\int_{\mathbb{R}}(1+|x|)\,|q(x,t)|\,dx<\infty,\qquad \text{for all } T>0, \tag{1.1}$$

for q together with its first three derivatives q_x, q_{xx}, and q_{xxx} with respect to x. For existence of solutions with such initial data we refer to Section 4.2 in [15].

Associated with the KdV equation is the following Lax-pair:

$$L(t) = -\partial_x^2 + q(.,t), \tag{1.2}$$

$$P(t) = -4\partial_x^3 + 6q(.,t)\partial_x + 3q_x(.,t), \tag{1.3}$$

and

$$\frac{d}{dt}L(t) = [P(t),L(t)] = P(t)L(t) - L(t)P(t) \text{ on } H^5(\mathbb{R}),$$

which is equivalent to the KdV equation. Moreover, $P(t)$ is skew adjoint with $\mathfrak{D}(P(t)) = H^3(\mathbb{R})$. We are more interested in the self-adjoint Schrödinger operator

$$L(t) = H(t) = -\frac{d^2}{dx^2} + q(.,t),\qquad \mathfrak{D}(H) = H^2(\mathbb{R})\subset L^2(\mathbb{R}). \tag{1.4}$$

$L^2(\mathbb{R})$ denotes the Hilbert space of square integrable (complex-valued) functions over $\mathbb{R}$. By our assumption (1.1) the spectrum of H consists of an absolutely continuous part $[0,\infty)$ plus a finite number of eigenvalues $-\kappa_j^2 \in (-\infty, 0]$, $1 \leq j \leq N$. In addition, there exist two Jost solutions $\psi_\pm(k,x,t)$, which solve the differential equation

$$H(t)\psi_\pm(k,x,t) = k^2\psi_\pm(k,x,t), \qquad \mathrm{Im}(k) > 0, \tag{1.5}$$

and asymptotically look like the free solutions

$$\lim_{x\to\pm\infty} \mathrm{e}^{\mp \mathrm{i}kx}\psi_\pm(k,x,t) = 1. \tag{1.6}$$

Both $\psi_\pm(k,x,t)$ are analytic for $\mathrm{Im}(k) > 0$ and continuous for $\mathrm{Im}(k) \geq 0$.

Theorem 1.1. *The asymptotics of the two Jost solutions are given by*

$$\psi_\pm(k,x,t) = \mathrm{e}^{\pm \mathrm{i}kx}\Big(1 + Q_\pm(x,t)\frac{1}{2\mathrm{i}k} + O(\frac{1}{k^2})\Big), \tag{1.7}$$

as $k \to \infty$ *with* $\mathrm{Im}(k) \geq 0$*, where*

$$Q_+(x,t) = -\int_x^\infty q(y,t)dy, \quad Q_-(x,t) = -\int_{-\infty}^x q(y,t)dy. \tag{1.8}$$

Proof. The Jost solutions $\psi_\pm(k,x,t)$ are the unique solutions of the following integral equation for all k with $\mathrm{Im}(k) \geq 0$

$$\psi_\pm(k,x,t) = \mathrm{e}^{\pm \mathrm{i}kx} - \int_x^{\pm\infty} \frac{\sin(k(x-y))}{k} q(y,t)\psi_\pm(k,y,t)dy. \tag{1.9}$$

We furthermore know that $\lim_{x\to\pm\infty} \mathrm{e}^{\mp \mathrm{i}kx}\psi_\pm(k,x,t) = 1 + O(k^{-1})$. We will proof the asymptotic only for $\psi_+(k,x,t)$:

$$\begin{aligned}
&\mathrm{e}^{-\mathrm{i}kx}\psi_+(k,x,t)\\
&= 1 + \int_x^\infty \frac{\mathrm{e}^{2\mathrm{i}k(y-x)} - 1}{2\mathrm{i}k} q(y,t)\mathrm{e}^{-\mathrm{i}ky}\psi_+(k,y,t)dy\\
&= 1 + \int_x^\infty \frac{\mathrm{e}^{2\mathrm{i}k(y-x)} - 1}{2\mathrm{i}k} q(y,t)dy + O\left(\frac{1}{k}\right)\int_x^\infty \frac{\mathrm{e}^{2\mathrm{i}k(x-y)} - 1}{2\mathrm{i}k} q(y,t)dy\\
&= 1 - \frac{1}{2\mathrm{i}k}\int_x^\infty q(y,t)dy + \int_x^\infty \frac{\mathrm{e}^{2\mathrm{i}k(y-x)}}{2\mathrm{i}k} q(y,t)dy + O\left(\frac{1}{k^2}\right)\\
&= 1 - \frac{1}{2\mathrm{i}k}\int_x^\infty q(y,t)dy + O\left(\frac{1}{k^2}\right)
\end{aligned}$$

as $k \to \infty$. For the first equality we used $\sin(k) = \frac{\mathrm{e}^{\mathrm{i}k} - \mathrm{e}^{-\mathrm{i}k}}{2\mathrm{i}}$. □

Considering the Wronskian $W(\overline{\psi_\pm}, \psi_\pm) = \overline{\psi_\pm}\psi_\pm' - \overline{\psi_\pm}'\psi_\pm$, where $'$ denotes the derivation with respect to x, we see that it is independent of x and t along the real axis. Thus we can compute $W(\overline{\psi_\pm}, \psi_\pm)(k) = \pm 2\mathrm{i}k$, which shows that these two functions are linearly independent, but the Schrödinger equation can have at most two linearly independent solutions, which in our case are represented by ψ_+ and ψ_-. Therefore we can write $\overline{\psi_\pm}$ as a linear combination of the given Jost solutions. In particular, one has the scattering relations

$$T(k)\psi_\mp(k,x,t) = \overline{\psi_\pm(k,x,t)} + R_\pm(k,t)\psi_\pm(k,x,t), \qquad k \in \mathbb{R}, \tag{1.10}$$

where $T(k)$, $R_\pm(k,t)$ are the transmission respectively reflection coefficients. The transmission and reflection coefficients have the following well-known properties:

Lemma 1.2. *The transmission coefficient $T(k)$ is meromorphic for $\mathrm{Im}(k) > 0$ with simple poles at $\mathrm{i}\kappa_1, \dots, \mathrm{i}\kappa_N$ and is continuous up to the real line. The residues of $T(k)$ are given by*

$$\mathrm{Res}_{\mathrm{i}\kappa_j} T(k) = \mathrm{i}\mu_j(t)\gamma_{+,j}(t)^2 = \mathrm{i}\mu_j\gamma_{+,j}^2, \tag{1.11}$$

where

$$\gamma_{+,j}(t)^{-1} = \|\psi_+(\mathrm{i}\kappa_j, ., t)\|_2 \tag{1.12}$$

and $\psi_+(\mathrm{i}\kappa_j, x, t) = \mu_j(t)\psi_-(\mathrm{i}\kappa_j, x, t)$.

Moreover,

$$T(k)\overline{R_+(k,t)} + \overline{T(k)}R_-(k,t) = 0, \qquad |T(k)|^2 + |R_\pm(k,t)|^2 = 1. \tag{1.13}$$

The functions $q(x,t)$, $x \in \mathbb{R}$ for fixed $t \in \mathbb{R}$ are uniquely determined by their right scattering data, that is, by the right reflection coefficient $R_+(k,t), k \in \mathbb{R}$ and the eigenvalues $\kappa_j \in (0,\infty), j = 1,\dots,N$, together with the corresponding norming constants $\gamma_{+,j}(t) > 0, j = 1,\dots,N$. So in particular, one reflection coefficient, say $R(k,t) = R_+(k,t)$, and one set of norming constants, say $\gamma_j(t) = \gamma_{+,j}(t)$, suffices. Moreover, the time dependence is given by:

Lemma 1.3. *The time evolutions of the quantities $R_+(k,t)$, $T(k,t)$ and $\gamma_{+,j}(t)$ are given by*

$$R(k,t) = R(k)\mathrm{e}^{8\mathrm{i}k^3 t}, \tag{1.14}$$

$$\gamma_j(t) = \gamma_j \mathrm{e}^{4\kappa_j^3 t}, \tag{1.15}$$

$$T(k,t) = T(k), \tag{1.16}$$

where $R(k) = R(k,0)$, $T(k) = T(k,0)$ and $\gamma_j = \gamma_j(0)$.

1.2 The Riemann-Hilbert problem for the KdV equation

We will define a Riemann–Hilbert problem as follows:

$$m(k,x,t) = \begin{cases} \begin{pmatrix} T(k)\psi_-(k,x,t)\mathrm{e}^{\mathrm{i}kx} & \psi_+(k,x,t)\mathrm{e}^{-\mathrm{i}kx} \end{pmatrix}, & \mathrm{Im}(k) > 0, \\ \begin{pmatrix} \psi_+(-k,x,t)\mathrm{e}^{\mathrm{i}kx} & T(-k)\psi_-(-k,x,t)\mathrm{e}^{-\mathrm{i}kx} \end{pmatrix}, & \mathrm{Im}(k) < 0. \end{cases} \tag{1.17}$$

We are interested in the jump condition of $m(k,x,t)$ on the real axis $\mathbb{R}$ (oriented from negative to positive). To formulate our jump condition we use the following convention: When representing functions on $\mathbb{R}$, the lower subscript denotes the non-tangential limit from different sides. By $m_+(k)$ we denote the limit from above and by $m_-(k)$ the one from below. Using the notation above implicitly assumes that these limits exist in the sense that $m(k)$ extends to a continuous function on the real axis away from 0.

Theorem 1.4 (Vector Riemann–Hilbert problem)**.** *Let $\mathcal{S}_+(H(0)) = \{R(k),\ k \geq 0;\ (\kappa_j, \gamma_j),\ 1 \leq j \leq N\}$ the right scattering data of the operator $H(0)$. Then $m(k) = m(k,x,t)$ defined in (1.17) is meromorphic away from the real axis with simple poles at $\mathrm{i}\kappa_j$, $-\mathrm{i}\kappa_j$ and satisfies:*

(i) The jump condition

$$m_+(k) = m_-(k)v(k), \qquad v(k) = \begin{pmatrix} 1-|R(k)|^2 & -\overline{R(k)}\mathrm{e}^{-t\Phi(k)} \\ R(k)\mathrm{e}^{t\Phi(k)} & 1 \end{pmatrix}, \tag{1.18}$$

for $k \in \mathbb{R}$,

(ii) the pole conditions

$$\begin{aligned} \operatorname{Res}_{\mathrm{i}\kappa_j} m(k) &= \lim_{k\to \mathrm{i}\kappa_j} m(k) \begin{pmatrix} 0 & 0 \\ \mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} & 0 \end{pmatrix}, \\ \operatorname{Res}_{-\mathrm{i}\kappa_j} m(k) &= \lim_{k\to -\mathrm{i}\kappa_j} m(k) \begin{pmatrix} 0 & -\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{1.19}$$

(iii) the symmetry condition

$$m(-k) = m(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1.20}$$

(iv) and the normalization

$$\lim_{\kappa\to\infty} m(\mathrm{i}\kappa) = \begin{pmatrix} 1 & 1 \end{pmatrix}. \tag{1.21}$$

Here the phase is given by

$$\Phi(k) = 8\mathrm{i}k^3 + 2\mathrm{i}k\frac{x}{t}. \tag{1.22}$$

Proof. (i) For the proof of the jump condition we need the scattering relations (1.10) and (1.13).

$$v(k)^{-1} = \begin{pmatrix} 1 & \overline{R(k)}\mathrm{e}^{-t\Phi(k)} \\ -R(k)\mathrm{e}^{t\Phi(k)} & 1-|R(k)|^2 \end{pmatrix} \tag{1.23}$$

and so we can also show $m_+(k)v(k)^{-1} = m_-(k)$.

$$\begin{aligned} &T(k)\psi_-(k,x,t)\mathrm{e}^{\mathrm{i}kx} - R(k)\mathrm{e}^{t\Phi(k)}\psi_+(k,x,t)\mathrm{e}^{-\mathrm{i}kx} \\ &= T(k)\psi_-(k,x,t)\mathrm{e}^{\mathrm{i}kx} - R(k,t)\mathrm{e}^{\mathrm{i}kx}\psi_+(k,x,t) \\ &= \overline{\psi_+(k,x,t)}\mathrm{e}^{\mathrm{i}kx} = \psi_+(-k,x,t)\mathrm{e}^{\mathrm{i}kx}, \end{aligned}$$

hence we have proven the jump condition for the first component of $m_-(k)$. For the second component of $m_-(k)$ we compute

$$\begin{aligned} &T(k)\psi_-(k,x,t)\mathrm{e}^{\mathrm{i}kx}\overline{R(k)}\mathrm{e}^{-t\Phi(k)} + (1-|R(k)|^2)\psi_+(k,x,t)\mathrm{e}^{-\mathrm{i}kx} \\ &= T(k)\psi_-(k,x,t)\mathrm{e}^{-\mathrm{i}kx}\overline{R(k,t)} + |T(k)|^2\,\psi_+(k,x,t)\mathrm{e}^{-\mathrm{i}kx} \\ &= \overline{\psi_+(k,x,t)}\mathrm{e}^{-\mathrm{i}kx}\overline{R(k,t)} + |R(k,t)|^2\,\mathrm{e}^{-\mathrm{i}kx}\psi_+(k,x,t) + |T(k)|^2\,\mathrm{e}^{-\mathrm{i}kx}\psi_+(k,x,t) \\ &= \overline{\psi_+(k,x,t)R(k,t)}\mathrm{e}^{-\mathrm{i}kx} + \psi_+(k,x,t)\mathrm{e}^{-\mathrm{i}kx} \\ &= \overline{T(k)\psi_-(k,x,t)}\mathrm{e}^{-\mathrm{i}kx} = T(-k)\psi_-(-k,x,t)\mathrm{e}^{-\mathrm{i}kx}. \end{aligned}$$

(ii) First of all note that the Jost solutions $\psi_\pm(k,x,t)$ are analytic for $\mathrm{Im}(k) > 0$ and that the transmission coefficient $T(k)$ has only simple poles at $\mathrm{i}\kappa_j$. Hence $\mathrm{Res}_{\mathrm{i}\kappa_j} m_2(k) = 0$ and $\mathrm{Res}_{-\mathrm{i}\kappa_j} m_1(k) = 0$. Moreover

$$\begin{aligned}\mathrm{Res}_{\mathrm{i}\kappa_j} m_1(k) &= \mathrm{Res}_{\mathrm{i}\kappa_j} T(k)\psi_-(\mathrm{i}\kappa_j,x,t)\mathrm{e}^{-\kappa_j x}\\ &= \mathrm{i}\mu_j(t)\gamma_j(t)^2\psi_-(\mathrm{i}\kappa_j,x,t)\mathrm{e}^{-\kappa_j x}\\ &= \mathrm{i}\gamma_j(t)^2\psi_+(\mathrm{i}\kappa_j,x,t)\mathrm{e}^{-\kappa_j x}\\ &= \mathrm{i}\gamma_j^2\mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}\psi_+(\mathrm{i}\kappa_j,x,t)\mathrm{e}^{\kappa_j x}.\end{aligned}$$

Similarly $\mathrm{Res}_{-\mathrm{i}\kappa_j} m_2(k)$ can be computed. Here $m_l(k)$ denotes the l'th component of $m(k)$.

(iii) The symmetry condition is obvious from the construction of our function $m(k,x,t)$.

(iv) The normalization follows immediately from the next Lemma

□

Remark 1.5. *Observe that the pole condition at $\mathrm{i}\kappa_j$ is sufficient since the one at $-\mathrm{i}\kappa_j$ follows by symmetry as the following calculation shows*

$$\begin{aligned}\mathrm{Res}_{-\mathrm{i}\kappa_j} m(k) &= \lim_{k\to-\mathrm{i}\kappa_j}(k+\mathrm{i}\kappa_j)m(k)\\ &= -\lim_{k\to\mathrm{i}\kappa_j}(k-\mathrm{i}\kappa_j)m(k)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ &= -\lim_{k\to\mathrm{i}\kappa_j} m(k)\begin{pmatrix}0 & 0\\ \mathrm{i}\gamma_j^2\mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ &= -\lim_{k\to\mathrm{i}\kappa_j} m(-k)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ \mathrm{i}\gamma_j^2\mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ &= \lim_{k\to-\mathrm{i}\kappa_j} m(k)\begin{pmatrix}0 & -\mathrm{i}\gamma_j^2\mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}\\ 0 & 0\end{pmatrix}.\end{aligned}$$

Moreover, we have the following asymptotic behavior as $k\to\infty$ with $\mathrm{Im}(k)\geq 0$:

Lemma 1.6. *The function $m(k,x,t)$ defined in (1.17) satisfies*

$$m(k,x,t) = \begin{pmatrix}1 & 1\end{pmatrix} + Q(x,t)\frac{1}{2\mathrm{i}k}\begin{pmatrix}-1 & 1\end{pmatrix} + O\left(\frac{1}{k^2}\right). \tag{1.24}$$

Here $Q(x,t) = Q_+(x,t)$ is defined in (1.8).

Proof. This follows from (1.7) and $T(k)\psi_-(k,x,t)\psi_+(k,x,t) = 1+\frac{q(x,t)}{2k^2}+O(\frac{1}{k^4})$. For details we refer to [12]. □

For our further analysis it will be convenient to rewrite the pole condition as a jump condition and hence turn our meromorphic Riemann–Hilbert problem into a holomorphic Riemann–Hilbert problem following [6]. Choose ε so small that the discs $|k - \mathrm{i}\kappa_j| < \varepsilon$ lie inside the upper half plane and do not intersect. Then redefine $m(k)$ in a neighborhood of $\mathrm{i}\kappa_j$ respectively $-\mathrm{i}\kappa_j$ according to

$$m(k) = \begin{cases} m(k) \begin{pmatrix} 1 & 0 \\ -\frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}{k-\mathrm{i}\kappa_j} & 1 \end{pmatrix}, & |k - \mathrm{i}\kappa_j| < \varepsilon, \\ m(k) \begin{pmatrix} 1 & \frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}{k+\mathrm{i}\kappa_j} \\ 0 & 1 \end{pmatrix}, & |k + \mathrm{i}\kappa_j| < \varepsilon, \\ m(k), & \text{else}. \end{cases} \tag{1.25}$$

Note that for $\mathrm{Im}(k) < 0$ we redefined $m(k)$ with respect to our symmetry (1.20). Then a straightforward calculation using $\mathrm{Res}_{\mathrm{i}\kappa}\, m(k) = \lim_{k\to \mathrm{i}\kappa}(k - \mathrm{i}\kappa)m(k)$ shows:

Lemma 1.7. *Suppose $m(k)$ is redefined as in (1.25). Then $m(k)$ is holomorphic away from the real axis and the small circles around $\mathrm{i}\kappa_j$ and $-\mathrm{i}\kappa_j$. Furthermore it satisfies (1.18), (1.20), (1.21) and the pole condition is replaced by the jump condition*

$$\begin{aligned} m_+(k) &= m_-(k) \begin{pmatrix} 1 & 0 \\ -\frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}{k-\mathrm{i}\kappa_j} & 1 \end{pmatrix}, \quad |k - \mathrm{i}\kappa_j| = \varepsilon, \\ m_+(k) &= m_-(k) \begin{pmatrix} 1 & -\frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}{k+\mathrm{i}\kappa_j} \\ 0 & 1 \end{pmatrix}, \quad |k + \mathrm{i}\kappa_j| = \varepsilon, \end{aligned} \tag{1.26}$$

where the small circle around $\mathrm{i}\kappa_j$ is oriented counterclockwise and the one around $-\mathrm{i}\kappa_j$ clockwise.

Proof. (1.18),(1.20), and (1.21) still hold, because $m(k)$ is only redefined with respect to our symmetry condition on the small circles around $\mathrm{i}\kappa_j, 1 \le j \le N$. A simple calculation inserting the definition of the new $m(k)$ and using the former pole condition shows that the pole condition is replaced by the jump condition (1.26). $\square$

Next we turn to uniqueness of the solution of this vector Riemann–Hilbert problem. This will also explain the reason for our symmetry condition. We begin by observing that if there is a point $k_1 \in \mathbb{C}$, such that $m(k_1) = \begin{pmatrix} 0 & 0 \end{pmatrix}$, then $n(k) = \frac{1}{k-k_1} m(k)$ satisfies the same jump and pole conditions as $m(k)$. However, it will clearly violate the symmetry condition! Hence, without the symmetry condition, the solution of our vector Riemann–Hilbert problem will not be unique in such a situation. Moreover, a look at the one soliton solution verifies that this case indeed can happen.

Lemma 1.8 (One soliton solution)**.** *Suppose there is only one eigenvalue and that the reflection coefficient vanishes, that is, $\mathcal{S}_+(H(t)) = \{R(k,t) \equiv 0,\ k \in \mathbb{R};\ (\kappa, \gamma(t)), \kappa > 0, \gamma > 0\}$. Then the unique solution of the Riemann–Hilbert problem (1.18)–(1.21) is given by*

$$\begin{aligned} m_0(k) &= \begin{pmatrix} f(k) & f(-k) \end{pmatrix} \\ f(k) &= \frac{1}{1 + (2\kappa)^{-1}\gamma^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa)}} \left(1 + \frac{k + \mathrm{i}\kappa}{k - \mathrm{i}\kappa} (2\kappa)^{-1} \gamma^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa)} \right). \end{aligned} \tag{1.27}$$

In particular,

$$Q_+(x,t) = \frac{2\gamma^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa)}}{1+(2\kappa)^{-1}\gamma^2\mathrm{e}^{t\Phi(\mathrm{i}\kappa)}}. \tag{1.28}$$

Proof. By assumption the reflection coefficient vanishes and so the jump along the real axis disappears. Therefore and by the symmetry condition, we know that the solution is of the form $m_0(k) = \begin{pmatrix} f(k) & f(-k) \end{pmatrix}$, where $f(k)$ is meromorphic. Furthermore the function $f(k)$ has only a simple pole at $\mathrm{i}\kappa$, so that we can make the ansatz $f(k) = C + D\frac{k+\mathrm{i}\kappa}{k-\mathrm{i}\kappa}$. Then the constants C and D are uniquely determined by the pole conditions and the normalization. □

In fact, observe $f(k_1) = f(-k_1) = 0$ if and only if $k_1 = 0$ and $2\kappa = \gamma^2\mathrm{e}^{t\Phi(\mathrm{i}\kappa)}$. Furthermore, even in the general case $m(k_1) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ can only occur at $k_1 = 0$ as the following lemma shows.

Lemma 1.9. *If $m(k_1) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ for m defined as in (1.17), then $k_1 = 0$. Moreover, the zero of at least one component is simple in this case.*

Proof. By (1.17) the condition $m(k_1) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ implies that the Jost solutions $\psi_-(k,x)$ and $\psi_+(k,x)$ are linearly dependent or that the transmission coefficient $T(k_1) = 0$. This can only happen, at the band edge, $k_1 = 0$ or at an eigenvalue $k_1 = \mathrm{i}\kappa_j$.

We begin with the case $k_1 = \mathrm{i}\kappa_j$. In this case the derivative of the Wronskian $W(k) = (\psi_+(k,x)\psi_-'(k,x) - \psi_+'(k,x)\psi_-(k,x))$ does not vanish by the well-known formula $\frac{d}{dk}W(k)|_{k=k_1} = -2k_1\int_{\mathbb{R}}\psi_+(k_1,x)\psi_-(k_1,x)dx \neq 0$. Moreover, the diagonal Green's function $g(z,x) = W(k)^{-1}\psi_+(k,x)\psi_-(k,x)$ is Herglotz as a function of $z = -k^2$ and hence can have at most a simple zero at $z = -k_1^2$. Since $z \to -k^2$ is conformal away from $z = 0$ the same is true as a function of k. Hence, if $\psi_+(\mathrm{i}\kappa_j,x) = \psi_-(\mathrm{i}\kappa_j,x) = 0$, both can have at most a simple zero at $k = \mathrm{i}\kappa_j$. But $T(k)$ has a simple pole at $\mathrm{i}\kappa_j$ and hence $T(k)\psi_-(k,x)$ cannot vanish at $k = \mathrm{i}\kappa_j$, a contradiction.

It remains to show that one zero is simple in the case $k_1 = 0$. In fact, one can show that $\frac{d}{dk}W(k)|_{k=k_1} \neq 0$ in this case as follows: First of all note that $\dot\psi_\pm(k)$ (where the dot denotes the derivative with respect to k) again solves $H\dot\psi_\pm(k_1) = -k_1^2\dot\psi_\pm(k_1)$ if $k_1 = 0$. Moreover, by $W(k_1) = 0$ we have $\psi_+(k_1) = c\,\psi_-(k_1)$ for some constant c (independent of x). Thus we can compute

$$\begin{aligned}\dot W(k_1) &= W(\dot\psi_+(k_1),\psi_-(k_1)) + W(\psi_+(k_1),\dot\psi_-(k_1))\\ &= c^{-1}W(\dot\psi_+(k_1),\psi_+(k_1)) + cW(\psi_-(k_1),\dot\psi_-(k_1))\end{aligned}$$

by letting $x \to +\infty$ for the first and $x \to -\infty$ for the second Wronskian (in which case we can replace $\psi_\pm(k)$ by $\mathrm{e}^{\pm\mathrm{i}kx}$), which gives

$$\dot W(k_1) = -\mathrm{i}(c + c^{-1}).$$

Hence the Wronskian has a simple zero. But if both functions had more than simple zeros, so would the Wronskian, a contradiction. □

Chapter 2

A uniqueness result for symmetric vector Riemann–Hilbert problems

In this chapter we want to investigate uniqueness for the holomorphic vector Riemann–Hilbert problem

$$\begin{aligned} &m_+(k) = m_-(k)v(k), \qquad k \in \Sigma, \\ &m(-k) = m(k)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ &\lim_{\kappa\to\infty} m(\mathrm{i}\kappa) = \begin{pmatrix} 1 & 1 \end{pmatrix}. \end{aligned} \tag{2.1}$$

Hypothesis H.2.1. *Let Σ consist of a finite number of smooth oriented curves in $\mathbb{C}$ such that the distance between Σ and $\{\mathrm{i}y | y \geq y_0\}$ is positive for some $y_0 > 0$. Assume that the contour Σ is invariant under $k \mapsto -k$ and $v(k)$ is symmetric*

$$v(-k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v(k)^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k \in \Sigma. \tag{2.2}$$

Moreover, suppose $\det(v(k)) = 1$.

Now we are ready to show that the symmetry condition in fact guarantees uniqueness.

Theorem 2.2. *Suppose there exists a solution $m(k)$ of the Riemann–Hilbert problem (2.1) for which $m(k) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ can happen at most for $k = 0$ in which case $\limsup_{k\to 0} \frac{k}{m_j(k)}$ is bounded from any direction for $j = 1$ or $j = 2$.*

Then the Riemann–Hilbert problem (2.1) with norming condition replaced by

$$\lim_{\kappa\to\infty} m(\mathrm{i}\kappa) = \begin{pmatrix} \alpha & \alpha \end{pmatrix} \tag{2.3}$$

for given $\alpha \in \mathbb{C}$, has a unique solution $m_\alpha(k) = \alpha\, m(k)$.

Proof. Let $m_\alpha(k)$ be a solution of (2.1) normalized according to (2.3). Then we can construct a matrix valued solution via $M = (m, m_\alpha)$ and there are two possible cases: Either $\det M(k)$ is nonzero for some k or it vanishes identically.

We start with the first case. By Lemma 1.7, we can rewrite all poles as jumps with determinant one. Hence, the determinant of this modified Riemann–Hilbert problem has no jump. Next consider a triangle, which intersects our contour Σ. We can now apply the same method as in the proof of the Schwarz reflection principle to conclude that the function $\det M(k)$ is holomorphic on $\mathbb{C}$. Since, it is bounded at infinity, we can apply Liouville's theorem and hence the determinant is constant. But taking determinants in

$$M(-k) = M(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

gives a contradiction.

It remains to investigate the case where $\det(M) \equiv 0$. In this case we have $m_\alpha(k) = \delta(k) m(k)$ with a scalar function δ. Moreover, $\delta(k)$ must be holomorphic for $k \in \mathbb{C}\backslash\Sigma$ and continuous for $k \in \Sigma$ except possibly at the points where $m(k_1) = \begin{pmatrix} 0 & 0 \end{pmatrix}$. Since it has no jump across Σ,

$$\delta_+(k) m_+(k) = m_{\alpha,+}(k) = m_{\alpha,-}(k) v(k) = \delta_-(k) m_-(k) v(k) = \delta_-(k) m_+(k),$$

we can conclude by the same method as in the first case that it is even holomorphic in $\mathbb{C}\backslash\{0\}$ with at most a simple pole at $k = 0$. Hence it must be of the form

$$\delta(k) = A + \frac{B}{k}.$$

Since δ has to be symmetric, $\delta(k) = \delta(-k)$, we obtain $B = 0$. Now, by the normalization, we obtain $\delta(k) = A = \alpha$. This finishes the proof. □

Furthermore, the requirements cannot be relaxed to allow (e.g.) second order zeros instead of simple zeros. In fact, if $m(k)$ is a solution for which both components vanish of second order at, say, $k = 0$, then $\tilde{m}(k) = \frac{1}{k^2} m(k)$ is a nontrivial symmetric solution of the vanishing problem (i.e. for $\alpha = 0$).

By Lemma 1.9 we have

Corollary 2.3. *The function $m(k, x, t)$ defined in (1.17) is the only solution of the vector Riemann–Hilbert problem (1.18)–(1.21).*

Proof. The function $m(k, x, t)$ defined in (1.17) satisfies the assumptions of Theorem 2.2 and therefore, is the unique solution of our Riemann-Hilbert problem. □

Observe that there is nothing special about $k \to \infty$ where we normalize, any other point would do as well. However, observe that for the one soliton solution (1.27), $f(k)$ vanishes at

$$k = \mathrm{i}\kappa \frac{1 - (2\kappa)^2 \gamma^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa)}}{1 + (2\kappa)^2 \gamma^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa)}}$$

and hence the Riemann–Hilbert problem normalized at this point has a nontrivial solution for $\alpha = 0$ and hence, by our uniqueness result, no solution for $\alpha = 1$. This shows that uniqueness and existence are connected, a fact which is not surprising since our Riemann–Hilbert problem is equivalent to a singular integral equation which is Fredholm of index zero (see Appendix B).

Chapter 3

Conjugation and deformation

This chapter demonstrates how to conjugate our Riemann-Hilbert problem and how to deform our jump contour, such that the jumps will be exponentially decreasing away from the stationary phase points. Furthermore the asymptotics in the soliton region are computed. We will further study how poles can be dealt with in this chapter, because solitons are represented in a Riemann-Hilbert problem by pole conditions.

3.1 Conjugation

For easy reference we note the following result:

Lemma 3.1 (Conjugation)**.** *Assume that $\widetilde{\Sigma} \subseteq \Sigma$. Let D be a matrix of the form*

$$D(k) = \begin{pmatrix} d(k)^{-1} & 0 \\ 0 & d(k) \end{pmatrix}, \tag{3.1}$$

where $d : \mathbb{C} \backslash \widetilde{\Sigma} \to \mathbb{C}$ is a sectionally analytic function. Set

$$\tilde{m}(k) = m(k) D(k), \tag{3.2}$$

then the jump matrix transforms according to

$$\tilde{v}(k) = D_-(k)^{-1} v(k) D_+(k). \tag{3.3}$$

If d satisfies $d(-k) = d(k)^{-1}$ and $d(k) = 1 + O(\frac{1}{|k|})$ as $k \to \infty$. Then the transformation $\tilde{m}(k) = m(k)D(k)$ respects our symmetry, that is, $\tilde{m}(k)$ satisfies (1.20) if and only if $m(k)$ does, and our normalization condition..

In particular, we obtain

$$\tilde{v} = \begin{pmatrix} v_{11} & v_{12} d^2 \\ v_{21} d^{-2} & v_{22} \end{pmatrix}, \qquad k \in \Sigma \backslash \widetilde{\Sigma}, \tag{3.4}$$

respectively

$$\tilde{v} = \begin{pmatrix} \frac{d_-}{d_+} v_{11} & v_{12} d_+ d_- \\ v_{21} d_+^{-1} d_-^{-1} & \frac{d_+}{d_-} v_{22} \end{pmatrix}, \qquad k \in \Sigma \cap \widetilde{\Sigma}. \tag{3.5}$$

Proof. For $k \in \Sigma$ we compute

$$\begin{aligned}\tilde{m}_+(k) &= m_+(k)D_+(k) = m_-(k)v(k)D_+(k) = m_-(k)D_-(k)D_-(k)^{-1}v(k)D_+(k)\\ &= \tilde{m}_-(k)\tilde{v}(k)\end{aligned}$$

and thus $\tilde{v}(k) = D_-(k)^{-1}v(k)D_+(k)$.

The symmetry condition follows by the next calculation

$$\begin{aligned}\tilde{m}(-k) &= m(-k)D(k)^{-1} = m(k)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}D(k)^{-1}\\ &= m(k)D(k)D(k)^{-1}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}D(k)^{-1} = \tilde{m}(k)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.\end{aligned}$$

□

In order to remove the poles there are two cases to distinguish. Some jumps are already exponentially decaying and in this case there is nothing to do.

Otherwise we use conjugation to turn the jumps into exponentially decaying ones, again following Deift, Kamvissis, Kriecherbauer, and Zhou [6]. It turns out that we will have to handle the poles at $\mathrm{i}\kappa_j$ and $-\mathrm{i}\kappa_j$ in one step in order to preserve symmetry and in order to not add additional poles elsewhere.

Lemma 3.2. *Assume that the Riemann–Hilbert problem for m has jump conditions near $\mathrm{i}\kappa$ and $-\mathrm{i}\kappa$ given by*

$$\begin{aligned}m_+(k) &= m_-(k)\begin{pmatrix}1 & 0\\ -\frac{\mathrm{i}\gamma^2}{k-\mathrm{i}\kappa} & 1\end{pmatrix}, \quad |k-\mathrm{i}\kappa| = \varepsilon,\\ m_+(k) &= m_-(k)\begin{pmatrix}1 & -\frac{\mathrm{i}\gamma^2}{k+\mathrm{i}\kappa}\\ 0 & 1\end{pmatrix}, \quad |k+\mathrm{i}\kappa| = \varepsilon.\end{aligned} \tag{3.6}$$

Then this Riemann–Hilbert problem is equivalent to a Riemann–Hilbert problem for $\tilde{m}$ which has jump conditions near $\mathrm{i}\kappa$ and $-\mathrm{i}\kappa$ given by

$$\begin{aligned}\tilde{m}_+(k) &= \tilde{m}_-(k)\begin{pmatrix}1 & -\frac{(k+\mathrm{i}\kappa)^2}{\mathrm{i}\gamma^2(k-\mathrm{i}\kappa)}\\ 0 & 1\end{pmatrix}, \qquad |k-\mathrm{i}\kappa| = \varepsilon,\\ \tilde{m}_+(k) &= \tilde{m}_-(k)\begin{pmatrix}1 & 0\\ -\frac{(k-\mathrm{i}\kappa)^2}{\mathrm{i}\gamma^2(k+\mathrm{i}\kappa)} & 1\end{pmatrix}, \qquad |k+\mathrm{i}\kappa| = \varepsilon,\end{aligned}$$

and all remaining data conjugated (as in Lemma 3.1) by

$$D(k) = \begin{pmatrix}\frac{k-\mathrm{i}\kappa}{k+\mathrm{i}\kappa} & 0\\ 0 & \frac{k+\mathrm{i}\kappa}{k-\mathrm{i}\kappa}\end{pmatrix}. \tag{3.7}$$

Proof. To turn γ^2 into γ^{-2}, introduce D by

$$D(k) = \begin{cases}\begin{pmatrix}1 & -\frac{k-\mathrm{i}\kappa}{\mathrm{i}\gamma^2}\\ \frac{\mathrm{i}\gamma^2}{k-\mathrm{i}\kappa} & 0\end{pmatrix}\begin{pmatrix}\frac{k-\mathrm{i}\kappa}{k+\mathrm{i}\kappa} & 0\\ 0 & \frac{k+\mathrm{i}\kappa}{k-\mathrm{i}\kappa}\end{pmatrix}, & |k-\mathrm{i}\kappa| < \varepsilon,\\ \begin{pmatrix}0 & -\frac{\mathrm{i}\gamma^2}{k+\mathrm{i}\kappa}\\ \frac{k+\mathrm{i}\kappa}{\mathrm{i}\gamma^2} & 1\end{pmatrix}\begin{pmatrix}\frac{k-\mathrm{i}\kappa}{k+\mathrm{i}\kappa} & 0\\ 0 & \frac{k+\mathrm{i}\kappa}{k-\mathrm{i}\kappa}\end{pmatrix}, & |k+\mathrm{i}\kappa| < \varepsilon,\\ \begin{pmatrix}\frac{k-\mathrm{i}\kappa}{k+\mathrm{i}\kappa} & 0\\ 0 & \frac{k+\mathrm{i}\kappa}{k-\mathrm{i}\kappa}\end{pmatrix}, & \text{else},\end{cases}$$

and note that $D(k)$ is analytic away from the two circles. Now set $\tilde{m}(k) = m(k)D(k)$. The new jump conditions can be verified with the same method as in the previous Lemma. □

3.2 The phase and the partial transmission coefficient

The jump along the real axis is of oscillatory type and our aim is to apply a contour deformation such that all jumps will be moved into regions where the oscillatory terms will decay exponentially. Since the jump matrix v contains both $\exp(t\Phi)$ and $\exp(-t\Phi)$ we need to separate them in order to be able to move them to different regions of the complex plane.

We recall that the phase of the associated Riemann–Hilbert problem is given by

$$\Phi(k) = 8\mathrm{i}k^3 + 2\mathrm{i}k\frac{x}{t} \tag{3.8}$$

and the stationary phase points, $\Phi'(k) = 0$, are denoted by

$$k_0 = \sqrt{-\frac{x}{12t}}, \quad -k_0 = -\sqrt{-\frac{x}{12t}}, \qquad \lambda_0 = \frac{x}{12t}. \tag{3.9}$$

For $\frac{x}{t} > 0$ we have $k_0 \in \mathrm{i}\mathbb{R}$, and for $\frac{x}{t} < 0$ we have $k_0 \in \mathbb{R}$. For $\frac{x}{t} > 0$ we will also need the value $\mathrm{i}\kappa_0 \in \mathrm{i}\mathbb{R}$ defined via $\mathrm{Re}(\Phi(\mathrm{i}\kappa_0)) = 0$, that is,

$$\frac{x}{t} = 4\kappa_0^2. \tag{3.10}$$

We will set $\kappa_0 = 0$ if $\frac{x}{t} < 0$ for notational convenience. A simple analysis shows that for $\frac{x}{t} > 0$ we have $0 < k_0/\mathrm{i} < \kappa_0$.

As mentioned above we will need the following factorization of the jump condition (1.18). The correct factorization for $\mathrm{Re}(k_0) < |k|$ is given by

$$v(k) = b_-(k)^{-1} b_+(k), \tag{3.11}$$

where

$$b_-(k) = \begin{pmatrix} 1 & \overline{R(k)}\mathrm{e}^{-t\Phi(k)} \\ 0 & 1 \end{pmatrix}, \qquad b_+(k) = \begin{pmatrix} 1 & 0 \\ R(k)\mathrm{e}^{t\Phi(k)} & 1 \end{pmatrix}.$$

For $|k| < \mathrm{Re}(k_0)$ the factorization is given by

$$v(k) = B_-(k)^{-1} \begin{pmatrix} 1 - |R(k)|^2 & 0 \\ 0 & \frac{1}{1-|R(k)|^2} \end{pmatrix} B_+(k), \tag{3.12}$$

where

$$B_-(k) = \begin{pmatrix} 1 & 0 \\ -\frac{R(k)\mathrm{e}^{t\Phi(k)}}{1-|R(k)|^2} & 1 \end{pmatrix}, \qquad B_+(k) = \begin{pmatrix} 1 & -\frac{\overline{R(k)}\mathrm{e}^{-t\Phi(k)}}{1-|R(k)|^2} \\ 0 & 1 \end{pmatrix}.$$

To get rid of the diagonal part in the factorization corresponding to $|k| < \mathrm{Re}(k_0)$ and to conjugate the jumps near the eigenvalues we need the partial transmission coefficient.

We define the partial transmission coefficient with respect to k_0 by

$$T(k,k_0) = \begin{cases} \prod\limits_{\kappa_j\in(\kappa_0,\infty)} \frac{k+\mathrm{i}\kappa_j}{k-\mathrm{i}\kappa_j}, & k_0\in\mathrm{i}\mathbb{R}^+, \\ \prod\limits_{j=1}^{N} \frac{k+\mathrm{i}\kappa_j}{k-\mathrm{i}\kappa_j} \exp\left(\frac{1}{2\pi\mathrm{i}} \int\limits_{-k_0}^{k_0} \frac{\log(|T(\zeta)|^2)}{\zeta-k} d\zeta\right), & k_0\in\mathbb{R}^+ \end{cases} \tag{3.13}$$

for $k\in\mathbb{C}\backslash\Sigma(k_0)$, where $\Sigma(k_0)=[-\mathrm{Re}(k_0),\mathrm{Re}(k_0)]$. Note that $T(k,k_0)$ can be computed in terms of the scattering data since $|T(k)|^2 = 1-|R_+(k,t)|^2$.

Moreover, we conclude that

$$T(k,k_0) = 1 + T_1(k_0)\frac{\mathrm{i}}{k} + O\left(\frac{1}{k^2}\right), \qquad \text{as } k\to\infty$$

where

$$T_1(k_0) = \begin{cases} \sum\limits_{\kappa_j\in(\kappa_0,\infty)} 2\kappa_j, & k_0\in\mathrm{i}\mathbb{R}^+, \\ \sum\limits_{\kappa_j\in(\kappa_0,\infty)} 2\kappa_j + \frac{1}{2\pi}\int_{-k_0}^{k_0}\log(|T(\zeta)|^2)d\zeta, & k_0\in\mathbb{R}^+. \end{cases}$$

For the next theorem, we need the Plemelj formula

Theorem 3.3 (Plemelj)**.** *Let L be a simple smooth oriented arc. If $\psi(t)$ is a function satisfying a Hölder condition on L, then*

$$\Psi_\pm(t) = \pm\frac{1}{2}\psi(t) + \frac{1}{2\pi\mathrm{i}}\int_L \frac{\psi(\tau)}{\tau-t}d\tau, \tag{3.14}$$

or, equivalently

$$\Psi_+(t) - \Psi_-(t) = \psi(t), \tag{3.15}$$

$$\Psi_+(t) + \Psi_-(t) = \frac{1}{\pi\mathrm{i}}\int_L \frac{\psi(\tau)}{\tau-t}d\tau. \tag{3.16}$$

Here the connection between ψ and Ψ is given by

$$\Psi(t) = \frac{1}{2\pi\mathrm{i}}\int_L \frac{\psi(\tau)}{\tau-t}d\tau. \tag{3.17}$$

Proof. A proof can be found in Muskhelishvili [16]. □

Theorem 3.4. *The partial transmission coefficient $T(k,k_0)$ is meromorphic in $\mathbb{C}\backslash\Sigma(k_0)$, where $\Sigma(k_0)=[-\mathrm{Re}(k_0),\mathrm{Re}(k_0)]$, with simple poles at $\mathrm{i}\kappa_j$ and simple zeros at $-\mathrm{i}\kappa_j$ for all j with $\kappa_0<\kappa_j$, and satisfies the jump condition*

$$T_+(k,k_0) = T_-(k,k_0)(1-|R(k)|^2), \qquad \text{for } k\in\Sigma(k_0). \tag{3.18}$$

Moreover,

(i) $T(-k,k_0) = T(k,k_0)^{-1}, k\in\mathbb{C}\backslash\Sigma(k_0)$ and $\lim_{k\to\infty} T(k,k_0) > 0$

(ii) $\overline{T(\overline{k},k_0)} = T(k,k_0)^{-1}$.

Proof. That $\mathrm{i}\kappa_j$ are simple poles and $-\mathrm{i}\kappa_j$ are simple zeros is obvious from the Blaschke factors and that $T(k,k_0)$ has the given jump follows from Plemelj formula. (i) and (ii) are straightforward to check. □

3.3 Deformation

Now we are ready to perform our conjugation step. Introduce

$$D(k) = \begin{cases} \begin{pmatrix} 1 & -\frac{k-\mathrm{i}\kappa_j}{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}} \\ \frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}{k-\mathrm{i}\kappa_j} & 0 \end{pmatrix} D_0(k), & |k-\mathrm{i}\kappa_j|<\varepsilon,\ \kappa_0<\kappa_j, \\ \begin{pmatrix} 0 & -\frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}{k+\mathrm{i}\kappa_j} \\ \frac{k+\mathrm{i}\kappa_j}{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}} & 1 \end{pmatrix} D_0(k), & |k+\mathrm{i}\kappa_j|<\varepsilon,\ \kappa_0<\kappa_j, \\ D_0(k), & \text{else}, \end{cases}$$

where

$$D_0(k) = \begin{pmatrix} T(k,k_0)^{-1} & 0 \\ 0 & T(k,k_0) \end{pmatrix}.$$

Note that we have

$$D(-k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now we conjugate our problem using $D(k)$ and set $\tilde{m}(k) = m(k)D(k)$.

Then using Lemma 3.1 and Lemma 3.2 the jump corresponding to $\kappa_0 < \kappa_j$ (if any) is given by

$$\begin{aligned} \tilde{v}(k) &= \begin{pmatrix} 1 & -\frac{k-\mathrm{i}\kappa_j}{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} T(k,k_0)^{-2}} \\ 0 & 1 \end{pmatrix}, \qquad |k-\mathrm{i}\kappa_j| = \varepsilon, \\ \tilde{v}(k) &= \begin{pmatrix} 1 & 0 \\ -\frac{k+\mathrm{i}\kappa_j}{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} T(k,k_0)^{2}} & 1 \end{pmatrix}, \qquad |k+\mathrm{i}\kappa_j| = \varepsilon, \end{aligned} \tag{3.19}$$

and corresponding to $\kappa_0 > \kappa_j$ (if any) by

$$\begin{aligned} \tilde{v}(k) &= \begin{pmatrix} 1 & 0 \\ -\frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} T(k,k_0)^{-2}}{k-\mathrm{i}\kappa_j} & 1 \end{pmatrix}, \qquad |k-\mathrm{i}\kappa_j| = \varepsilon, \\ \tilde{v}(k) &= \begin{pmatrix} 1 & -\frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} T(k,k_0)^{2}}{k+\mathrm{i}\kappa_j} \\ 0 & 1 \end{pmatrix}, \qquad |k+\mathrm{i}\kappa_j| = \varepsilon. \end{aligned} \tag{3.20}$$

In particular, all jumps corresponding to poles, except for possibly one if $\kappa_j = \kappa_0$, are exponentially decreasing. In this case we will keep the pole condition which now reads

$$\begin{aligned} \operatorname{Res}_{\mathrm{i}\kappa_j} \tilde{m}(k) &= \lim_{k\to \mathrm{i}\kappa_j} \tilde{m}(k) \begin{pmatrix} 0 & 0 \\ \mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} T(\mathrm{i}\kappa_j,k_0)^{-2} & 0 \end{pmatrix}, \\ \operatorname{Res}_{-\mathrm{i}\kappa_j} \tilde{m}(k) &= \lim_{k\to -\mathrm{i}\kappa_j} \tilde{m}(k) \begin{pmatrix} 0 & -\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} T(\mathrm{i}\kappa_j,k_0)^{-2} \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.21}$$

Furthermore, the jump along $\mathbb{R}$ is given by

$$\tilde{v}(k) = \begin{cases} \tilde{b}_-(k)^{-1}\tilde{b}_+(k), & \operatorname{Re}(k_0) < |k|, \\ \tilde{B}_-(k)^{-1}\tilde{B}_+(k), & \operatorname{Re}(k_0) > |k|, \end{cases} \tag{3.22}$$

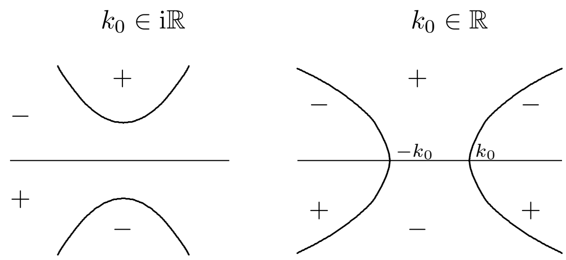

Figure 3.1: Sign of $\mathrm{Re}(\Phi(k))$ for different values of k_0

where

$$\tilde{b}_-(k) = \begin{pmatrix} 1 & \frac{R(-k)\mathrm{e}^{-t\Phi(k)}}{T(-k,k_0)^2} \\ 0 & 1 \end{pmatrix}, \quad \tilde{b}_+(k) = \begin{pmatrix} 1 & 0 \\ \frac{R(k)\mathrm{e}^{t\Phi(k)}}{T(k,k_0)^2} & 1 \end{pmatrix}, \tag{3.23}$$

and

$$\tilde{B}_-(k) = \begin{pmatrix} 1 & 0 \\ -\frac{T_-(k,k_0)^{-2}}{1-|R(k)|^2} R(k)\mathrm{e}^{t\Phi(k)} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{T_-(-k,k_0)}{T_-(k,k_0)} R(k)\mathrm{e}^{t\Phi(k)} & 1 \end{pmatrix},$$

$$\tilde{B}_+(k) = \begin{pmatrix} 1 & -\frac{T_+(k,k_0)^2}{1-|R(k)|^2} R(-k)\mathrm{e}^{-t\Phi(k)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{T_+(k,k_0)}{T_+(-k,k_0)} R(-k)\mathrm{e}^{-t\Phi(k)} \\ 0 & 1 \end{pmatrix}.$$

Here we have used

$$T_\pm(-k, k_0) = T_\mp(k, k_0)^{-1}, \qquad k \in \Sigma(k_0)$$

and the jump condition for the partial transmission coefficient $T(k, k_0)$ along $\Sigma(k_0)$ in the last step, which shows that the matrix entries are bounded for $k \in \mathbb{R}$.

Note also that we have used $T(k, k_0)^{-1} = \overline{T(k, k_0)}$ and $R(-k) = \overline{R(k)}$ for $k \in \mathbb{R}$ to show that there exists an analytic continuation into the neighborhood of the real axis.

Now we deform the jump along $\mathbb{R}$ to move the oscillatory terms into regions where they are decaying. There are two cases to distinguish:

Case 1: $k_0 \in \mathrm{i}\mathbb{R}$, $k_0 \neq 0$:

We set $\Sigma_\pm = \{k \in \mathbb{C} | \mathrm{Im}(k) = \pm\varepsilon\}$ for some small ε such that $\Sigma_\pm$ lies in the region with $\pm\mathrm{Re}(k) < 0$ and such that the circles around $\pm\mathrm{i}\kappa_j$ lie outside the region in between Σ_- and Σ_+. Then we can split our jump by redefining $\tilde{m}(k)$ according to

$$\widehat{m}(k) = \begin{cases} \tilde{m}(k)\tilde{b}_+(k)^{-1}, & 0 < \mathrm{Im}(k) < \varepsilon, \\ \tilde{m}(k)\tilde{b}_-(k)^{-1}, & -\varepsilon < \mathrm{Im}(k) < 0, \\ \tilde{m}(k), & \text{else}. \end{cases} \tag{3.24}$$

Thus the jump along the real axis disappears and the jump along $\Sigma_\pm$ is given by

$$\widehat{v}(k) = \begin{cases} \tilde{b}_+(k), & k \in \Sigma_+ \\ \tilde{b}_-(k)^{-1}, & k \in \Sigma_-. \end{cases} \tag{3.25}$$

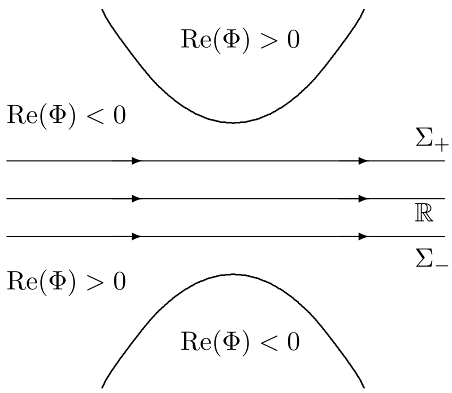

Figure 3.2: Deformed contour for $k_0 \in \mathrm{i}\mathbb{R}^+$

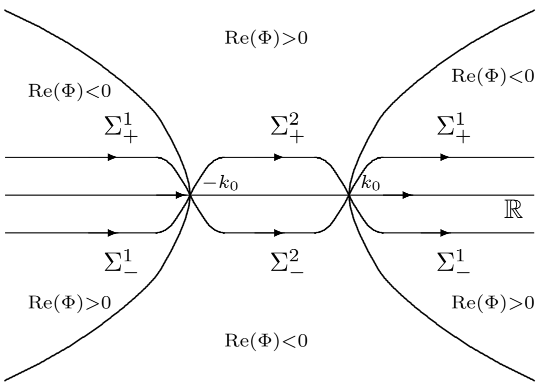

Figure 3.3: Deformed contour for $k_0 \in \mathbb{R}^+$

All other jumps are unchanged. Note that the resulting Riemann-Hilbert problem still satisfies our symmetry condition (1.20), since we have

$$\tilde{b}_\pm(-k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{b}_\mp(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.26}$$

By construction the jump along $\Sigma_\pm$ is exponentially decreasing as $t \to \infty$.

Case 2: $k_0 \in \mathbb{R}$, $k_0 \neq 0$:

We set $\Sigma_\pm = \Sigma_\pm^1 \cup \Sigma_\pm^2$ according to Figure 3.3 chosen such that the circles around $\pm\mathrm{i}\kappa_j$ lie outside the region in between Σ_- and Σ_+. Again note that $\Sigma_\pm^1$ respectively $\Sigma_\pm^2$ lie in the region with $\pm\mathrm{Re}(\Phi(k)) < 0$. Then we can split our jump by redefining $\tilde{m}(k)$ according to

$$\widehat{m}(k) = \begin{cases} \tilde{m}(k)\tilde{b}_+(k)^{-1}, & k \text{ between } \mathbb{R} \text{ and } \Sigma_+^1, \\ \tilde{m}(k)\tilde{b}_-(k)^{-1}, & k \text{ between } \mathbb{R} \text{ and } \Sigma_-^1, \\ \tilde{m}(k)\tilde{B}_+(k)^{-1}, & k \text{ between } \mathbb{R} \text{ and } \Sigma_+^2, \\ \tilde{m}(k)\tilde{B}_-(k)^{-1}, & k \text{ between } \mathbb{R} \text{ and } \Sigma_-^2, \\ \tilde{m}(k), & \text{else}. \end{cases} \tag{3.27}$$

One checks that the jump along $\mathbb{R}$ disappears and the jump along $\Sigma_\pm$ is given by

$$\widehat{v}(k) = \begin{cases} \tilde{b}_+(k), & k \in \Sigma_+^1, \\ \tilde{b}_-(k)^{-1}, & k \in \Sigma_-^1, \\ \tilde{B}_+(k), & k \in \Sigma_+^2, \\ \tilde{B}_-(k)^{-1}, & k \in \Sigma_-^2. \end{cases} \tag{3.28}$$

All other jumps are unchanged. Again the resulting Riemann-Hilbert problem still satisfies our symmetry condition (1.20) and the jump along $\Sigma_\pm \backslash \{k_0, -k_0\}$ is exponentially decreasing as $t \to \infty$

3.4 The long-time asymptotics in the soliton region

Now we are ready to state and proof one of our main results:

Theorem 3.5. *Assume (1.1) and abbreviate by $c_j = 4\kappa_j^2$ the velocity of the j'th soliton determined by $\mathrm{Re}(\Phi(\mathrm{i}\kappa_j)) = 0$. Then the asymptotics in the soliton region, $x/t \geq C$ for some $C > 0$, are as follows:*

Let $\varepsilon > 0$ sufficiently small such that the intervals $[c_j - \varepsilon, c_j + \varepsilon]$, $1 \leq j \leq N$, are disjoint and lie inside $\mathbb{R}^+$.

If $|\frac{x}{t} - c_j| < \varepsilon$ for some j, one has

$$\int_x^\infty q(y,t)dy = \frac{-2\gamma_j^2(x,t)}{1 + (2\kappa_j)^{-1}\gamma_j^2(x,t)} - 2T_1(\mathrm{i}\frac{\kappa_j}{\sqrt{3}}) + O(t^{-l}),$$

respectively

$$q(x,t) = \frac{-4\kappa_j\gamma_j^2(x,t)}{(1 + (2\kappa_j)^{-1}\gamma_j^2(x,t))^2} + O(t^{-l}) \tag{3.29}$$

for any $l \geq 1$, where

$$\gamma_j^2(x,t) = \gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} T(\mathrm{i}\kappa_j, \mathrm{i}\frac{\kappa_j}{\sqrt{3}})^{-2}. \tag{3.30}$$

If $|\frac{x}{t} - c_j| \geq \varepsilon$, for all j, one has

$$\int_x^\infty q(y,t)dy = -2T_1(k_0) + O(t^{-l}),$$

respectively

$$q(x,t) = O(t^{-l}) \tag{3.31}$$

for any $l \geq 1$.

Proof. $\widehat{m}$ has the following asymptotic

$$\begin{aligned}\widehat{m}(k) &= m_0 + m_1\frac{1}{k} + O\left(\frac{1}{k^2}\right)\\ &= \begin{pmatrix} T(-k,k_0) & T(k,k_0)\end{pmatrix}\\ &+ \begin{pmatrix} -T(-k,k_0)Q_+(x,t)(2\mathrm{i}k)^{-1} & T(k,k_0)Q_+(x,t)(2\mathrm{i}k)^{-1}\end{pmatrix} + O\left(\frac{1}{k^2}\right)\\ &= \begin{pmatrix} 1 & 1\end{pmatrix} + \begin{pmatrix} -1 & 1\end{pmatrix}\frac{\mathrm{i}T_1(k_0)}{k} + \begin{pmatrix} -1 & 1\end{pmatrix}\frac{Q_+(x,t)}{2\mathrm{i}k} + O\left(\frac{1}{k^2}\right).\end{aligned}$$

Thus we have

$$m_1 = \begin{pmatrix} -1 & 1\end{pmatrix}\left(\mathrm{i}T_1(k_0) + \frac{Q_+(x,t)}{2\mathrm{i}}\right). \tag{3.32}$$

By construction the jump along $\Sigma_\pm$ is exponentially decreasing as $t \to \infty$. Hence we can apply Theorem B.7 as follows:

If $\left|\frac{x}{t} - c_j\right| > \varepsilon$ (resp. $|\kappa_0^2 - \kappa_j^2|$) for all j we can choose $\gamma_0 = 0$ and $w_0^t \equiv 0$ by removing all jumps corresponding to poles from w^t. The error between the solutions of w_0^t and w^t is exponentially small in the sense of Theorem B.7. In particular $\|w^t - w_0^t\|_\infty \le O(t^{-l})$ as $t \to \infty$ for all $l \ge 1$ and $\|w^t - w_0^t\|_2 \le O(t^{-l})$ as $t \to \infty$ for all $l \ge 1$ and so the associated Riemann-Hilbert problems only differ by $O(t^{-l})$. ¿From Lemma 1.8, we have the one soliton solution $\widehat{m}_0 = \begin{pmatrix}\widehat{f}(k) & \widehat{f}(-k)\end{pmatrix}$ with $\widehat{f}(k) \equiv 1$ for $|\mathrm{Im}(k)|$ big enough and so $Q_+(x,t) = +2T_1(k_0) + O(t^{-l})$.

If $\left|\frac{x}{t} - c_j\right| < \varepsilon$ (resp. $|\kappa_0^2 - \kappa_j^2|$) for some j, we choose $\gamma_0^t = \gamma_k(x,t)$ and $w_0^t \equiv 0$. As before we conclude that the error between the solutions of w^t and w_0^t is exponentially small in the sense of Theorem B.7 and so the associated solutions of the Riemann-Hilbert problems only differ by $O(t^{-l})$. ¿From Lemma 1.8, we have the one soliton solution $\widehat{m}_0 = \begin{pmatrix}\widehat{f}(k) & \widehat{f}(-k)\end{pmatrix}$ with $\widehat{f}(k) = \frac{1}{1+(2\kappa_j)^{-1}\gamma_j^2(x,t)}(1 + \frac{k+\mathrm{i}\kappa_j}{k-\mathrm{i}\kappa_j}(2\kappa_j)^{-1}\gamma_j^2(x,t))$ for $|\mathrm{Im}(k)|$ big enough where

$$\gamma_j^2(x,t) = \gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} T(\mathrm{i}\kappa_j, \mathrm{i}\frac{\kappa_j}{\sqrt{3}})^{-2}$$

and hence, plugging in the power series expansion,

$$Q_+(x,t) = +2T_1(\mathrm{i}\frac{\kappa_j}{\sqrt{3}}) + \frac{2\gamma_j^2(x,t)}{1+(2\kappa_j)^{-1}\gamma_j^2(x,t)} + O(t^{-l}). \tag{3.33}$$

For the second part recall from Lemma 1.6 that

$$T(k)\psi_+(k,x,t)\psi_-(k,x,t) = 1 + \frac{q(x,t)}{2k^2} + O(\frac{1}{k^4}). \tag{3.34}$$

In the first case $\left|\frac{x}{t} - c_j\right| > \varepsilon$ for all j we know that the solution $\widehat{m}(k)$ is of the form

$$\widehat{m}(k) = \begin{pmatrix} 1 & 1\end{pmatrix} + O(t^{-l})$$

and we can therefore conclude by multiplying the two components that

$$q(x,t) = O(t^{-l}). \tag{3.35}$$

In the second case $\left|\frac{x}{t} - c_j\right| < \varepsilon$ for some j we know from the first part that the solution $\widehat{m}(k)$ is of the form

$$\widehat{m}(k) = \begin{pmatrix} \widehat{f}(k) & \widehat{f}(-k) \end{pmatrix} + O(t^{-l}),$$

where $\widehat{f}(k)$ is defined as in the first part. Multiplying the first and the second component and plugging in the power series expansion we get

$$\begin{aligned} q(x,t) &= \frac{-4\kappa_j \gamma_j^2(x,t)}{1 + (2\kappa_j)^{-1}\gamma_j^2(x,t)} + \frac{2\gamma_j^4(x,t)}{(1 + (2\kappa_j)^{-1}\gamma_j^2(x,t))^2} + O(t^{-l}) \\ &= \frac{-4\kappa_j \gamma_j^2(x,t)}{(1 + (2\kappa_j)^{-1}\gamma_j^2(x,t))^2} + O(t^{-l}). \end{aligned}$$

□

Corollary 3.6. *Assume (1.1), then the asymptotic in the soliton region, $x/t \geq C$ for some $C > 0$, is as follows*

$$q(x,t) = \sum_{j=1}^{N} \frac{-4\kappa_j \gamma_j^2(x,t)}{(1 + (2\kappa_j)^{-1}\gamma_j^2(x,t))^2} + O(t^{-l}), \tag{3.36}$$

where $\gamma_j^2(x,t) = \gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} T(\mathrm{i}\kappa_j, \mathrm{i}\frac{\kappa_j}{\sqrt{3}})^{-2}$.

Proof. The claim follows immediately after some easy calculations:

(i) If $\left|\frac{x}{t} - c_j\right| < \varepsilon$ for some j, we can use the last theorem to obtain the corresponding term.

(ii) If the eigenvalue κ_j lies in the region where $\mathrm{Re}(\Phi(k)) < 0$, we see that $\gamma_j^2(x,t)$ is exponentially decreasing as $t \to \infty$ and therefore the whole, corresponding term is exponentially decreasing as $t \to \infty$.

(iii) If the eigenvalue κ_j lies in the region where $\mathrm{Re}(\Phi(k)) > 0$, we look at the corresponding term:

$$\frac{-4\kappa_j \gamma_j^2(x,t)}{(1 + (2\kappa_j)^{-1}\gamma_j^2(x,t))^2} = -\frac{16\kappa_j^3 \gamma_j^{-2}(x,t)}{(1 + (2\kappa_j)\gamma_j^{-2}(x,t))^2}. \tag{3.37}$$

Thus these term is also exponentially decaying as $t \to \infty$, because $\gamma_j^{-2}(x,t)$ is exponentially decreasing.

This finishes the proof. □

Remark 3.7. *This is exactly the same result as mentioned in the introduction as the following calculation shows:*

$$\begin{aligned} q(x,t) &= \sum_{j=1}^{N} \frac{-4\kappa_j \gamma_j^2(x,t)}{(1 + (2\kappa_j)^{-1}\gamma_j^2(x,t))^2} \\ &= -2\sum_{j=1}^{N} \frac{4\kappa_j^2}{\mathrm{e}^{-t\Phi(\mathrm{i}\kappa_j) - 2p_j} + \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j) + 2p_j} + 2} \\ &= -2\sum_{j=1}^{N} \frac{\kappa_j^2}{\cosh^2(\kappa_j x - 4\kappa_j^3 t - p_j)}, \end{aligned} \tag{3.38}$$

where we used that

$$T(\mathrm{i}\kappa_j, \mathrm{i}\frac{\kappa_j}{\sqrt{3}}) = \prod_{\kappa_l \in (\kappa_j, \infty)} \frac{\kappa_j + \kappa_l}{\kappa_j - \kappa_l}. \tag{3.39}$$

Chapter 4

The Riemann-Hilbert problem in the similarity region

In the previous section we have seen that for $k_0 \in \mathbb{R}$ we can reduce everything to a Riemann-Hilbert problem for $\widehat{m}(k)$ such that the jumps are exponentially decaying except in small neighborhoods of the stationary phase points k_0 and $-k_0$. Hence we need to continue our investigation of this case in this chapter.

4.1 Decoupling

Denote by $\Sigma^c(\pm k_0)$ the parts of $\Sigma_+ \cup \Sigma_-$ inside a small neighborhood of $\pm k_0$. We will now show that solving the two problems on the small crosses $\Sigma^c(k_0)$ respectively $\Sigma^c(-k_0)$ will lead us to the solution of our original problem.

Theorem 4.1 (Decoupling)**.** *Consider the Riemann–Hilbert problem*

$$\begin{aligned} m_+(k) = m_-(k)v(k), \qquad k \in \Sigma, \\ m(\infty) = \begin{pmatrix} 1 & 1 \end{pmatrix}, \end{aligned} \tag{4.1}$$

and let $0 < \alpha < \beta \le 2\alpha, \rho(t) \to \infty$ be given.

Suppose that for every sufficiently small $\varepsilon > 0$ both the L^2 and the L^∞ norms of v are $O(t^{-\beta})$ away from some ε neighborhoods of some points $k_j, 1 \le j \le n$. Moreover, suppose that the solution of the matrix problem with jump $v(k)$ restricted to the ε neighborhood of k_j has a solution which satisfies

$$M_j(k) = \mathbb{I} + \frac{1}{\rho(t)^\alpha}\frac{M_j}{k - k_j} + O(\rho(t)^{-\beta}), \qquad |k - k_j| > \varepsilon. \tag{4.2}$$

Then the solution $m(k)$ is given by

$$m(k) = \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{\rho(t)^\alpha}\begin{pmatrix} 1 & 1 \end{pmatrix}\sum_{j=1}^{n}\frac{M_j}{k - k_j} + O(\rho(t)^{-\beta}), \tag{4.3}$$

where the error term depends on the distance of k to Σ.

Proof. In this proof we will use the theory developed in Appendix B with $m_0(k) = \mathbb{I}$ and the usual Cauchy kernel $\Omega_\infty(s,k) = \mathbb{I}\frac{ds}{s-k}$. Assume that $m(k)$ exists. Introduce $\tilde{m}(k)$ by

$$\tilde{m}(k) = \begin{cases} m(k)M_j(k)^{-1}, & |k-k_j| \le 2\varepsilon, \\ m(k), & \text{else}. \end{cases} \tag{4.4}$$

The Riemann–Hilbert problem for $\tilde{m}(k)$ has jumps given by

$$\tilde{v}(k) = \begin{cases} M_j(k)^{-1}, & |k-k_j| = 2\varepsilon, \\ M_j(k)v(k)M_j(k)^{-1}, & k \in \Sigma, \varepsilon < |k-k_j| < 2\varepsilon, \\ \mathbb{I}, & k \in \Sigma, |k-k_j| < \varepsilon, \\ v(k), & \text{else}. \end{cases} \tag{4.5}$$

By assumption the jumps are $\mathbb{I} + O(\rho(t)^{-\alpha})$ on the circles $|k-k_j| = 2\varepsilon$ and even $\mathbb{I} + O(\rho(t)^{-\beta})$ on the rest (both in L^2 and L^∞ norms). In particular as in Lemma A.4 we infer

$$\left\|\tilde{\mu} - \begin{pmatrix} 1 & 1 \end{pmatrix}\right\|_2 = O(\rho(t)^{-\alpha}). \tag{4.6}$$

Thus we can conclude

$$\begin{aligned} m(k) &= \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{2\pi\mathrm{i}} \int_{\tilde{\Sigma}} \tilde{\mu}(s)\tilde{w}(s)\frac{ds}{s-k} \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{2\pi\mathrm{i}} \sum_{j=1}^n \int_{|s-k_j|=\varepsilon} \tilde{\mu}(s)(M_j(s)^{-1} - \mathbb{I})\frac{ds}{s-k} + O(\rho(t)^{-\beta}) \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} - \rho(t)^{-\alpha} \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{2\pi\mathrm{i}} \sum_{j=1}^n M_j \int_{|s-k_j|=\varepsilon} \frac{1}{s-k_j}\frac{ds}{s-k} + O(\rho(t)^{-\beta}) \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} + \rho(t)^{-\alpha} \begin{pmatrix} 1 & 1 \end{pmatrix} \sum_{j=1}^n \frac{M_j}{k-k_j} + O(\rho(t)^{-\beta}), \end{aligned} \tag{4.7}$$

and hence the claim is proven. ∎

4.2 The long-time asymptotics in the similarity region

Now let us turn to the solution of the problem on $\Sigma^c(k_0) = (\Sigma_+ \cup \Sigma_-) \cap \{k \,|\, |k-k_0| < \varepsilon\}$ for some small $\varepsilon > 0$. Since, we no longer impose the symmetry condition, we can also deform our contour slightly such that $\Sigma^c(k_0)$ consists of two straight lines. Next,

$$\Phi(k_0) = -16\mathrm{i}k_0^3, \qquad \Phi''(k_0) = 48\mathrm{i}k_0.$$

As a first step we make a change of coordinates

$$\zeta = \sqrt{48k_0}(k-k_0), \qquad k = k_0 + \frac{\zeta}{\sqrt{48k_0}} \tag{4.8}$$

such that the phase reads $\Phi(k) = \Phi(k_0) + \frac{\mathrm{i}}{2}\zeta^2 + O(\zeta^3)$. The corresponding Riemann-Hilbert problem will be solved in Appendix A. To apply this result, we will need the behavior of our jump matrix near k_0, that is, the behavior of $T(k, k_0)$ near k_0.

Lemma 4.2. *Let $k_0 \in \mathbb{R}$, then*

$$T(k, k_0) = \left(\frac{k - k_0}{k + k_0}\right)^{\mathrm{i}\nu} \tilde{T}(k, k_0), \tag{4.9}$$

where $\nu = -\frac{1}{\pi}\log(|T(k_0)|)$ and the branch cut of the logarithm is chosen along the negative real axis. Here

$$\tilde{T}(k, k_0) = \prod_{j=1}^{N} \frac{k + \mathrm{i}\kappa_j}{k - \mathrm{i}\kappa_j} \exp\left(\frac{1}{2\pi\mathrm{i}} \int_{-k_0}^{k_0} \log\left(\frac{|T(\zeta)|^2}{|T(k_0)|^2}\right) \frac{1}{\zeta - k} d\zeta\right) \tag{4.10}$$

is Hölder continuous of any exponent less then 1 at the stationary phase point $k = k_0$ and satisfies $\tilde{T}(k_0, k_0) \in \mathbb{T}$.

Proof. First of all note that

$$\exp\left(\frac{1}{2\pi\mathrm{i}} \int_{-k_0}^{k_0} \log(|T(k_0)|^2) \frac{1}{\zeta - k} d\zeta\right) = \left(\frac{k - k_0}{k + k_0}\right)^{\mathrm{i}\nu}. \tag{4.11}$$

Moreover we know from Theorem 3.4

$$\left|\tilde{T}(k, k_0)\right| = \overline{\tilde{T}(k, k_0)}\tilde{T}(k, k_0) = \tilde{T}(\overline{k}, k_0)^{-1}\tilde{T}(k, k_0) = \tilde{T}(-\overline{k}, k_0)\tilde{T}(k, k_0) \tag{4.12}$$

for $k \in \mathbb{C}\backslash\Sigma(k_0)$. Furthermore the Blaschke products $\prod_{j=1}^{N} \frac{k+\mathrm{i}\kappa_j}{k-\mathrm{i}\kappa_j}$ are continuous for $k \neq \mathrm{i}\kappa_j, j = 1, \dots, N$ and

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}} \int_{-k_0}^{k_0} \log\left(\frac{|T(\zeta)|^2}{|T(k_0)|^2}\right) \frac{1}{\zeta - k} d\zeta + \frac{1}{2\pi\mathrm{i}} \int_{-k_0}^{k_0} \log\left(\frac{|T(\zeta)|^2}{|T(k_0)|^2}\right) \frac{1}{\zeta + \overline{k}} d\zeta \\ = \frac{k - \overline{k}}{2\pi\mathrm{i}} \int_{k_0}^{k_0} \log\left(\frac{|T(\zeta)|^2}{|T(k_0)|^2}\right) \frac{1}{|\zeta - k|^2} d\zeta, \end{aligned}$$

which tends to 0 as $k \to k_0$ with $k \in \mathbb{C}\backslash\Sigma(k_0)$. Hence $\left|\tilde{T}(k, k_0)\right| \to 1$ as $k \to k_0$ and so $\tilde{T}(k, k_0) \in \mathbb{T}$.

For the proof of the Hölder continuity of any exponent less than 1 at $k = k_0$, we refer to Muskhelishvili [16]. $\square$

If $k(\zeta)$ is defined as in (4.8) and $0 < \alpha < 1$, then there is an $L > 0$ such that

$$\left|T(k(\zeta), k_0) - \zeta^{\mathrm{i}\nu}\tilde{T}(k_0, k_0)\mathrm{e}^{-\mathrm{i}\nu\log(2k_0\sqrt{48k_0})}\right| \leq L\,|\zeta|^{\alpha}\,, \tag{4.13}$$

where the branch cut of $\zeta^{\mathrm{i}\nu}$ is the negative real axis. Here we used the following observations:

(i)

$$\left(\frac{k-k_0}{k+k_0}\right)^{\mathrm{i}\nu} = \zeta^{\mathrm{i}\nu}\left(\frac{1}{2k_0\sqrt{48k_0}+\zeta}\right)^{\mathrm{i}\nu} = \zeta^{\mathrm{i}\nu}F(\zeta,k_0). \tag{4.14}$$

This function $F(\zeta, k_0)$ is Hölder continuous of any exponent less than 1 at $\zeta = 0$.

(ii) With the same idea as for differentiable functions at $\zeta = 0$, we can show that the product of two Hölder continuous functions is again Hölder continuous.

We also have

$$|R(k(\zeta)) - R(k_0)| \le L\,|\zeta|^\alpha \tag{4.15}$$

and thus the assumptions of Theorem A.1 are satisfied with

$$r = R(k_0)\tilde{T}(k_0,k_0)^{-2}\mathrm{e}^{2\mathrm{i}\nu\log(2k_0\sqrt{48k_0})}. \tag{4.16}$$

Therefore we can conclude that the solution on $\Sigma^c(k_0)$ is given by

$$\begin{aligned} M_1^c(k) &= \mathbb{I} + \frac{1}{\zeta}\frac{\mathrm{i}}{t^{1/2}}\begin{pmatrix}0 & -\beta\\ \overline{\beta} & 0\end{pmatrix} + O(t^{-\alpha}) \\ &= \mathbb{I} + \frac{1}{\sqrt{48k_0}(k-k_0)}\frac{\mathrm{i}}{t^{1/2}}\begin{pmatrix}0 & -\beta\\ \overline{\beta} & 0\end{pmatrix} + O(t^{-\alpha}), \end{aligned} \tag{4.17}$$

where β is given by

$$\begin{aligned} \beta &= \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4-\arg(r)+\arg(\Gamma(\mathrm{i}\nu)))}\mathrm{e}^{-t\Phi(k_0)}t^{-\mathrm{i}\nu} \\ &= \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4-\arg(R(k_0))+\arg(\Gamma(\mathrm{i}\nu)))}\tilde{T}(k_0,k_0)^2(192k_0^3)^{-\mathrm{i}\nu}\mathrm{e}^{-t\Phi(k_0)}t^{-\mathrm{i}\nu}. \end{aligned} \tag{4.18}$$

and $1/2 < \alpha < 1$.

We also need the solution $M_2^c(k)$ on $\Sigma^c(-k_0)$. We make the following ansatz, which is inspired by the symmetry condition for the vector Riemann–Hilbert problem outside the two small crosses:

$$M_2^c(k) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} M_1^c(-k)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{4.19}$$

¿From this we conclude

$$M_2^c(k) = \mathbb{I} - \frac{1}{\sqrt{48k_0}(k+k_0)}\frac{\mathrm{i}}{t^{1/2}}\begin{pmatrix}0 & \overline{\beta}\\ -\beta & 0\end{pmatrix} + O(t^{-\alpha}). \tag{4.20}$$

Applying Theorem 4.1 leads to

$$\widehat{m}(k) = \begin{pmatrix}1 & 1\end{pmatrix} + \frac{1}{\sqrt{48k_0}}\frac{\mathrm{i}}{t^{1/2}}\left(\frac{1}{k-k_0}\begin{pmatrix}\overline{\beta} & -\beta\end{pmatrix} - \frac{1}{k+k_0}\begin{pmatrix}-\beta & \overline{\beta}\end{pmatrix}\right) + O(t^{-\alpha}). \tag{4.21}$$

We are now ready to state and proof our second main result:

Theorem 4.3. *Assume (1.1) with $l = 5$, then the asymptotics in the similarity region, $x/t \le -C$ for some $C > 0$, are given by*

$$\int_x^\infty q(y,t)dy = -2T_1(k_0) - \frac{\sqrt{\nu}}{\sqrt{3k_0}} \frac{1}{t^{1/2}} \cos(16tk_0^3 - \nu \log(192tk_0^3) + \delta(k_0)) + O(t^{-\alpha}) \tag{4.22}$$

respectively

$$q(x,t) = \sqrt{\frac{4\nu k_0}{3t}} \sin(16tk_0^3 - \nu \log(192tk_0^3) + \delta(k_0)) + O(t^{-\alpha}) \tag{4.23}$$

for any $1/2 < \alpha < 1$. Here

$$\nu = -\frac{1}{\pi} \log(|T(k_0)|), \tag{4.24}$$

$$\delta(k_0) = \frac{\pi}{4} - \arg(R(k_0)) + \arg(\Gamma(\mathrm{i}\nu)) + 2\arg(\tilde{T}(k_0,k_0)), \tag{4.25}$$

$$\tilde{T}(k_0,k_0) = \prod_{j=1}^N \frac{k_0 + \mathrm{i}\kappa_j}{k_0 - \mathrm{i}\kappa_j} \exp\left(\frac{1}{2\pi\mathrm{i}} \int_{-k_0}^{k_0} \log\left(\frac{|T(\zeta)|^2}{|T(k_0)|^2} \right) \frac{1}{\zeta - k_0} d\zeta \right). \tag{4.26}$$

Proof. As in the proof of the asymptotics in the soliton region we set $\widehat{m}(k) = m_0 + m_1 \frac{1}{k} + O(k^{-2})$ as $k \to \infty$ and conclude

$$m_1 = \begin{pmatrix} -1 & 1 \end{pmatrix} \left(\mathrm{i}T_1(k_0) + \frac{Q_+(x,t)}{2\mathrm{i}} \right). \tag{4.27}$$

Therefore we compute

$$\begin{aligned}
&\widehat{m}(k) \\
&= \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{\sqrt{48k_0}} \frac{\mathrm{i}}{t^{1/2}} \left(\frac{1}{k - k_0} \begin{pmatrix} \overline{\beta} & -\beta \end{pmatrix} - \frac{1}{k + k_0} \begin{pmatrix} -\beta & \overline{\beta} \end{pmatrix} \right) + O(t^{-\alpha}) \\
&= \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{\sqrt{48k_0}} \frac{\mathrm{i}}{t^{1/2}} \frac{1}{k} \left(\sum_{l=0}^\infty \left(\frac{k_0}{k} \right)^l \begin{pmatrix} \overline{\beta} & -\beta \end{pmatrix} - \sum_{l=0}^\infty \left(-\frac{k_0}{k} \right)^l \begin{pmatrix} -\beta & \overline{\beta} \end{pmatrix} \right) \\
&+ O(t^{-\alpha}),
\end{aligned}$$

which leads to

$$Q_+(x,t) = 2T_1(k_0) + \frac{4}{\sqrt{48k_0}} \frac{1}{t^{1/2}} (\mathrm{Re}(\beta)) + O(t^{-\alpha}), \tag{4.28}$$

with

$$\beta = \sqrt{\nu} \mathrm{e}^{\mathrm{i}(\pi/4 - \arg(R(k_0)) + \arg(\Gamma(\mathrm{i}\nu)))} \tilde{T}(k_0,k_0)^2 (192k_0^3)^{-\mathrm{i}\nu} \mathrm{e}^{-t\Phi(k_0)} t^{-\mathrm{i}\nu}. \tag{4.29}$$

Using the fact that $|\beta/\sqrt{\nu}| = 1$ proves the first claim. For the second part recall from Lemma 1.6 that

$$T(k)\psi_+(k,x,t)\psi_-(k,x,t) = 1 + \frac{q(x,t)}{2k^2} + O(\frac{1}{k^4}) = \widehat{m}_1 \widehat{m}_2. \tag{4.30}$$

Plugging in the power series expansions of the components of $\widehat{m}$ yields

$$q(x,t) = \sqrt{\frac{4k_0}{3t}}\mathrm{Im}(\beta). \tag{4.31}$$

$\square$

Remark 4.4. *The result mentioned in the introduction is the same as the one stated above, because*

$$\arg(\tilde{T}(k_0,k_0)) = 2\sum_{j=1}^{N} \arctan\left(\frac{\kappa_j}{k_0}\right) - \frac{1}{2\pi}\int_{-k_0}^{k_0} \log\left(\frac{|T(\zeta)|^2}{|T(k_0)|^2}\right)\frac{1}{\zeta - k_0} d\zeta$$

and

$$\begin{aligned}
&\frac{1}{\pi}\int_{-k_0}^{k_0} \log|\zeta - k_0|\, d\log(1 - |R(\zeta)|^2) \\
&= \lim_{k_1 \to k_0} \frac{1}{\pi}\int_{-k_0}^{k_1} \log|\zeta - k_0|\, d\log(1 - |R(\zeta)|^2 \\
&= \lim_{k_1 \to k_0} \frac{1}{\pi}\left(\log|\zeta - k_0| \log(1 - |R(\zeta)|^2)|_{-k_0}^{k_1} - \int_{-k_0}^{k_1} \frac{1}{\zeta - k_0}\log(1 - |R(\zeta)|^2) d\zeta\right) \\
&= \lim_{k_1 \to k_0} \frac{1}{\pi}\left(\log(|R(k_0)|^2)\int_{-k_0}^{k_1} \frac{1}{\zeta - k_0} d\zeta - \int_{-k_0}^{k_1} \frac{1}{\zeta - k_0}\log(1 - |R(\zeta)|^2) d\zeta\right) \\
&= -\frac{1}{\pi}\int_{-k_0}^{k_0} \log\left(\frac{1 - |R(\zeta)|^2}{1 - |R(k_0)|^2}\right)\frac{1}{\zeta - k_0} d\zeta = -\frac{1}{\pi}\int_{-k_0}^{k_0} \log\left(\frac{|T(\zeta)|^2}{|T(k_0)|^2}\right)\frac{1}{\zeta - k_0} d\zeta.
\end{aligned}$$

Remark 4.5. *Formally the equation (4.23) for q can be obtained by differentiating the equation (4.22) for Q with respect to x. That this step is admissible could be shown as in Deift and Zhou [10], however our approach avoids this step.*

Remark 4.6. *Note that Theorem 4.1 does not require uniform boundedness of the associated integral operator in contradistinction to Theorem B.7. We only need the knowledge of the solution in some small neighborhoods. However it cannot be used in the soliton region, because our solution is not of the form $\mathbb{I} + o(1)$.*

Chapter 5

Analytic Approximation

In this chapter we want to present the necessary changes in the case where the reflection coefficient does not have an analytic extension. The idea is to use an analytic approximation and to split the reflection in an analytic part plus a small rest. The analytic part will be moved to the complex plane while the rest remains on the real axis. This needs to be done in such a way that the rest is of $O(t^{-1})$ and the growth of the analytic part can be controlled by the decay of the phase.

In the soliton region a straightforward splitting based on the Fourier transform

$$R(k) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx} \hat{R}(x) dx. \tag{5.1}$$

will be sufficient.

If our solution $q(x,t)$ is decaying rapid enough, we can conclude that $\hat{R} \in L^1(\mathbb{R})$ and furthermore $x^l \hat{R}(-x) \in L^1(0,\infty)$. For details we refer to [15].

Lemma 5.1. *Suppose $\hat{R} \in L^1(\mathbb{R})$, $x^l \hat{R}(-x) \in L^1(0,\infty)$ and let $\varepsilon, \beta > 0$ be given. Then we can split the reflection coefficient according to $R(k) = R_{a,t}(k) + R_{r,t}(k)$ such that $R_{a,t}(k)$ is analytic in $0 < \mathrm{Im}(k) < \varepsilon$ and*

$$|R_{a,t}(k)\mathrm{e}^{-\beta t}| = O(t^{-l}), \quad 0 < \mathrm{Im}(k) < \varepsilon, \qquad |R_{r,t}(k)| = O(t^{-l}), \quad k \in \mathbb{R}. \tag{5.2}$$

Proof. We choose $R_{a,t}(k) = \int_{-K(t)}^{\infty} \mathrm{e}^{\mathrm{i}kx} \hat{R}(x) dx$ with $K(t) = \frac{\beta_0}{\varepsilon} t$ for some positive $\beta_0 < \beta$. Then, for $0 < \mathrm{Im}(k) < \varepsilon$,

$$\begin{aligned} \left|R_{a,t}(k)\mathrm{e}^{-\beta t}\right| &= \mathrm{e}^{-\beta t} \left| \int_{-K(t)}^{\infty} \mathrm{e}^{\mathrm{i}kx} \hat{R}(x) dx \right| \le \mathrm{e}^{-\beta t} \int_{-K(t)}^{\infty} |\hat{R}(x)| \mathrm{e}^{-\mathrm{Im}(k)x} dx \\ &\le \mathrm{e}^{-\beta t} \mathrm{e}^{K(t)\mathrm{Im}(k)} \|\hat{R}\|_1 \le \mathrm{e}^{-\beta t} \mathrm{e}^{K(t)\varepsilon_1} \|\hat{R}\|_1 \\ &\le \|\hat{R}\|_1 \mathrm{e}^{-(\beta - \beta_0)t}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \|\hat{R}\|_1 &= \int_{\mathbb{R}} |\hat{R}(x)| dx = \int_{\mathbb{R}} (1+|x|)^{-1} (1+|x|) |\hat{R}(x)| dx \\ &\le \|(1+|x|)^{-1}\|_2 \|(1+|x|)\hat{R}(x)\|_2 < \infty, \end{aligned} \tag{5.3}$$

which proves the first claim. Similarly, for $\mathrm{Im}(k) = 0$,

$$\begin{aligned} |R_{r,t}(k)| &\le \int_{-\infty}^{-K(t)} |\hat{R}(x)| dx = \int_{K(t)}^{\infty} \frac{x^{l+1}|\hat{R}(-x)|}{x^{l+1}} dx \\ &\le \frac{\left\| x^l \hat{R}(-x) \right\|_{L^1(0,\infty)}}{K(t)^l} \le \frac{const}{t^l}. \end{aligned} \tag{5.4}$$

□

To apply this lemma in the soliton region $k_0 \in \mathrm{i}\mathbb{R}^+$ we choose

$$\beta = \min_{\mathrm{Im}(k)=-\varepsilon} -\mathrm{Re}(\Phi(k)) > 0. \tag{5.5}$$

and split $R(k) = R_{a,t}(k) + R_{r,t}(k)$ according to Lemma 5.1 to obtain

$$\tilde{b}_\pm(k) = \tilde{b}_{a,t,\pm}(k)\tilde{b}_{r,t,\pm}(k) = \tilde{b}_{r,t,\pm}(k)\tilde{b}_{a,t,\pm}(k).$$

Here $\tilde{b}_{a,t,\pm}(k)$, $\tilde{b}_{r,t,\pm}(k)$ denote the matrices obtained from $\tilde{b}_\pm(k)$ as defined in (3.23) by replacing $R(k)$ with $R_{a,t}(k)$, $R_{r,t}(k)$, respectively. Now we can move the analytic parts into the complex plane as in Chapter 3 while leaving the rest on the real axis. Hence, rather then (3.25), the jump now reads

$$\hat{v}(k) = \begin{cases} \tilde{b}_{a,t,+}(k), & k \in \Sigma_+, \\ \tilde{b}_{a,t,-}(k)^{-1}, & k \in \Sigma_-, \\ \tilde{b}_{r,t,-}(k)^{-1}\tilde{b}_{r,t,+}(k), & k \in \mathbb{R}. \end{cases} \tag{5.6}$$

By construction we have $\hat{v}(k) = \mathbb{I} + O(t^{-l})$ on the whole contour and the rest follows as in Section 3.

In the similarity region not only $\tilde{b}_\pm$ occur as jump matrices but also $\tilde{B}_\pm$. These matrices $\tilde{B}_\pm$ have at first sight more complicated off diagonal entries, but a closer look shows that they have indeed the same form. As we will rewrite them in terms of the left rather then the right scattering data, we will use the following notations: $R_r(k) \equiv R_+(k)$ for the right and $R_l(k) \equiv R_-(k)$ for the left reflection coefficient. Moreover, let $T_r(k, k_0) \equiv T(k, k_0)$ respectively $T_l(k, k_0) \equiv T(k)/T(k, k_0)$ be the right respectively left partial transmission coefficient.

With this notation we have

$$\tilde{v}(k) = \begin{cases} \tilde{b}_-(k)^{-1}\tilde{b}_+(k), & \mathrm{Re}(k_0) < |k|, \\ \tilde{B}_-(k)^{-1}\tilde{B}_+(k), & \mathrm{Re}(k_0) > |k|, \end{cases} \tag{5.7}$$

where

$$\tilde{b}_-(k) = \begin{pmatrix} 1 & \frac{R_r(-k)\mathrm{e}^{-t\Phi(k)}}{T_r(-k,k_0)^2} \\ 0 & 1 \end{pmatrix}, \quad \tilde{b}_+(k) = \begin{pmatrix} 1 & 0 \\ \frac{R_r(k)\mathrm{e}^{t\Phi(k)}}{T_r(k,k_0)^2} & 1 \end{pmatrix},$$

and

$$\begin{aligned} \tilde{B}_-(k) &= \begin{pmatrix} 1 & 0 \\ -\frac{T_{r,-}(k,k_0)^{-2}}{|T(k)|^2} R_r(k)\mathrm{e}^{t\Phi(k)} & 1 \end{pmatrix}, \\ \tilde{B}_+(k) &= \begin{pmatrix} 1 & -\frac{T_{r,+}(k,k_0)^2}{|T(k)|^2} R_r(-k)\mathrm{e}^{-t\Phi(k)} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Using (1.13) we can further write

$$\tilde{B}_-(k) = \begin{pmatrix} 1 & 0 \\ \frac{R_l(-k)\mathrm{e}^{t\Phi(k)}}{T_l(-k,k_0)^2} & 1 \end{pmatrix}, \quad \tilde{B}_+(k) = \begin{pmatrix} 1 & \frac{R_l(k)\mathrm{e}^{-t\Phi(k)}}{T_l(k,k_0)^2} \\ 0 & 1 \end{pmatrix}. \tag{5.8}$$

In the similarity region we need to take the small vicinities of the stationary phase points into account. Since the phase is cubic near these points, we cannot use it to dominate the exponential growth of the analytic part away from the unit circle. Hence we will take the phase as a new variable and use the Fourier transform with respect to this new variable. Since this change of coordinates is singular near the stationary phase points, there is a price we have to pay, namely, requiring additional smoothness for $R(k)$. If our solution $q(x,t)$ is decaying rapidly enough then we can conclude $R(k) \in C^l(\mathbb{R})$. Therefore we begin with

Lemma 5.2. *Suppose $R(k) \in C^5(\mathbb{R})$. Then we can split $R(k)$ according to*

$$R(k) = R_0(k) + (k-k_0)(k+k_0)H(k), \qquad k \in \Sigma(k_0), \tag{5.9}$$

where $R_0(k)$ is a real rational function in k such that $H(k)$ vanishes at k_0, $-k_0$ of order three and has a Fourier transform

$$H(k) = \int_{\mathbb{R}} \hat{H}(x)\mathrm{e}^{x\Phi(k)}dx, \tag{5.10}$$

with $x\hat{H}(x)$ integrable.

Proof. We can construct a rational function, which satisfies $f_n(-k) = \overline{f_n(k)}$ for $k \in \mathbb{R}$, by making the ansatz $f_n(k) = \frac{k_0^{2n+4}+1}{k^{2n+4}+1}\sum_{j=0}^n \frac{1}{j!(2k_0)^j}(\alpha_j + \mathrm{i}\beta_j\frac{k}{k_0})(k-k_0)^j(k+k_0)^j$. Furthermore we can choose α_j, $\beta_j \in \mathbb{R}$ for $j = 1, \dots, n$, such that we can match the values of R and its first four derivatives at k_0, $-k_0$ at $f_n(k)$. Thus we will set $R_0(k) = f_4(k)$, with $\alpha_0 = \mathrm{Re}(R(k_0))$, $\beta_0 = \mathrm{Im}(R(k_0))$ and so on. Note that $R_0(k)$ is integrable. Hence $H(k) \in C^4(\mathbb{R})$ and vanishes together with its first three derivatives at k_0, $-k_0$.

Note that $\Phi(k)/\mathrm{i} = 8(k^3 - 3k_0^2k)$ is a polynomial of order three which has a maximum at $-k_0$ and a minimum at k_0. Thus the phase $\Phi(k)/\mathrm{i}$ restricted to $\Sigma(k_0)$ gives a one to one coordinate transform $\Sigma(k_0) \to [\Phi(k_0)/\mathrm{i}, \Phi(-k_0)/\mathrm{i}] = [-16k_0^3, 16k_0^3]$ and we can hence express $H(k)$ in this new coordinate (setting it equal to zero outside this interval). The coordinate transform locally looks like a cube root near k_0 and $-k_0$, however, due to our assumption that H vanishes there, H is still C^2 in this new coordinate and the Fourier transform with respect to this new coordinates exists and has the required properties. □

Moreover, as in Lemma 5.1 we obtain:

Lemma 5.3. *Let $H(k)$ be as in the previous lemma. Then we can split $H(k)$ according to $H(k) = H_{a,t}(k) + H_{r,t}(k)$ such that $H_{a,t}(k)$ is analytic in the region* $\mathrm{Re}(\Phi(k)) < 0$ *and*

$$|H_{a,t}(k)\mathrm{e}^{\Phi(k)t/2}| = O(1),\ \mathrm{Re}(\Phi(k)) < 0, \mathrm{Im}(k) \le 0, \quad |H_{r,t}(k)| = O(t^{-1}),\ k \in \mathbb{R}. \tag{5.11}$$

Proof. We choose $H_{a,t}(k) = \int_{-K(t)}^{\infty} \hat{H}(x) \mathrm{e}^{x\Phi(k)} dx$ with $K(t) = t/2$. Then we can conclude as in Lemma 5.1:

$$\begin{aligned} |H_{a,t}(k)\mathrm{e}^{\Phi(k)t/2}| &\le \int_{-K(t)}^{\infty} |\hat{H}(x)\mathrm{e}^{x\Phi(k)+\Phi(k)t/2}| dx \le \|\hat{H}(x)\|_1 |\mathrm{e}^{-K(t)\Phi(k)+\Phi(k)t/2}| \\ &\le \|\hat{H}(x)\|_1 |\mathrm{e}^{-\Phi(k)t/2+\Phi(k)t/2}| = \|\hat{H}(x)\|_1 \le const \end{aligned}$$

and

$$|H_{r,t}(k)| \le \int_{-\infty}^{-K(t)} |\hat{H}(x)| dx \le const \sqrt{\int_{-\infty}^{-K(t)} \frac{1}{x^4} dx} \le const \frac{1}{K(t)^{3/2}} \le const \frac{1}{t}.$$

□

By construction $R_{a,t}(k) = R_0(k) + (k - k_0)(k + k_0) H_{a,t}(k)$ will satisfy the required Lipschitz estimate in a vicinity of the stationary phase points (uniformly in t) and all jumps will be $\mathbb{I} + O(t^{-1})$. Hence we can proceed as in Chapter 4.

Appendix A

The Riemann-Hilbert problem on a small cross

In this chapter, which is taken from Krüger and Teschl [13], we will solve the Riemann–Hilbert problem on a small cross.

Introduce the cross $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_4$ (see Figure A.1) by

$$\begin{aligned} \Sigma_1 &= \{u \mathrm{e}^{-\mathrm{i}\pi/4},\, u \in [0,\infty)\} & \Sigma_2 &= \{u \mathrm{e}^{\mathrm{i}\pi/4},\, u \in [0,\infty)\} \\ \Sigma_3 &= \{u \mathrm{e}^{3\mathrm{i}\pi/4},\, u \in [0,\infty)\} & \Sigma_4 &= \{u \mathrm{e}^{-3\mathrm{i}\pi/4},\, u \in [0,\infty)\}. \end{aligned} \tag{A.1}$$

Orient Σ such that the real part of z increases in the positive direction. Denote by $\mathbb{D} = \{z,\, |z| < 1\}$ the open unit disc. Throughout this section $z^{\mathrm{i}\nu}$ will denote the function $\mathrm{e}^{\mathrm{i}\nu \log(z)}$, where the branch cut of the logarithm is chosen along the negative real axis $(-\infty, 0)$.

Now consider the Riemann–Hilbert problem given by

$$\begin{aligned} m_+(z) &= m_-(z) v_j(z), & z &\in \Sigma_j, \quad j = 1,2,3,4, \\ m(z) &\to \mathbb{I}, & z &\to \infty, \end{aligned} \tag{A.2}$$

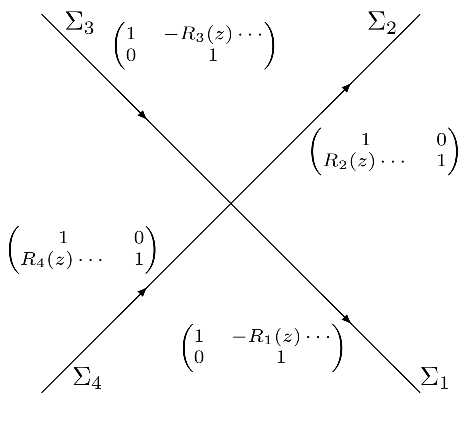

Figure A.1: Contours of a cross

where the jump matrices are given as follows: (v_j for $z \in \Sigma_j$)

$$v_1 = \begin{pmatrix} 1 & -R_1(z) z^{2\mathrm{i}\nu} \mathrm{e}^{-t\Phi(z)} \\ 0 & 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 & 0 \\ R_2(z) z^{-2\mathrm{i}\nu} \mathrm{e}^{t\Phi(z)} & 1 \end{pmatrix},$$
$$v_3 = \begin{pmatrix} 1 & -R_3(z) z^{2\mathrm{i}\nu} \mathrm{e}^{-t\Phi(z)} \\ 0 & 1 \end{pmatrix}, \qquad v_4 = \begin{pmatrix} 1 & 0 \\ R_4(z) z^{-2\mathrm{i}\nu} \mathrm{e}^{t\Phi(z)} & 1 \end{pmatrix}. \tag{A.3}$$

We can now state the next theorem, which gives us the solution of the Riemann–Hilbert problem (A.2). In the proof we follow the computations of section 3 and 4 in Deift and Zhou [9].

We will allow some variation, in all parameters as indicated.

Theorem A.1. *There is some $\rho_0 > 0$ such that $v_j(z) = \mathbb{I}$ for $|z| > \rho_0$. Moreover, suppose that within $|z| \le \rho_0$ the following estimates hold:*

(i) The phase satisfies $\Phi(0) \in \mathrm{i}\mathbb{R}$, $\Phi'(0) = 0$, $\Phi''(0) = \mathrm{i}$ and

$$\pm \mathrm{Re}\big(\Phi(z) - \Phi(0)\big) \ge \frac{1}{4}|z|^2, \qquad \begin{cases} + & \text{for } z \in \Sigma_1 \cup \Sigma_3, \\ - & \text{else,} \end{cases} \tag{A.4}$$

$$|\Phi(z) - \Phi(0) - \frac{\mathrm{i}z^2}{2}| \le C|z|^3. \tag{A.5}$$

(ii) There is some $r \in \mathbb{D}$ and constants $(\alpha, L) \in (0,1] \times (0, \infty)$ such that R_j, $j = 1, \dots, 4$, satisfy Hölder conditions of the form

$$|R_1(z) - \overline{r}| \le L|z|^\alpha, \qquad |R_2(z) - r| \le L|z|^\alpha,$$
$$|R_3(z) - \frac{\overline{r}}{1-|r|^2}| \le L|z|^\alpha, \qquad |R_4(z) - \frac{r}{1-|r|^2}| \le L|z|^\alpha. \tag{A.6}$$

Then the solution of the Riemann–Hilbert problem (A.2) satisfies

$$m(z) = \mathbb{I} + \frac{1}{z}\frac{\mathrm{i}}{t^{1/2}} \begin{pmatrix} 0 & -\beta \\ \overline{\beta} & 0 \end{pmatrix} + O(t^{-\frac{1+\alpha}{2}}), \tag{A.7}$$

for $|z| > \rho_0$, where

$$\beta = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(r) + \arg(\Gamma(\mathrm{i}\nu)))} \mathrm{e}^{-t\Phi(0)} t^{-\mathrm{i}\nu}, \qquad \nu = -\frac{1}{2\pi}\log(1 - |r|^2). \tag{A.8}$$

Furthermore, if $R_j(z)$ and $\Phi(z)$ depend on some parameter, the error term is uniform with respect to this parameter as long as r remains within a compact subset of $\mathbb{D}$ and the constants in the above estimates can be chosen independent of the parameters.

Remark A.2. *Note that the solution of the Riemann–Hilbert problem (A.2) is unique. This follows from the usual Liouville argument [5, Lem. 7.18] since* $\det(v_j) = 1$.

The proof will be given in the rest of this chapter, but split into a few parts for a better overview.

A.1 Approximation

A close look at the stated theorem shows, that the actual value of ρ_0 is of no importance. In fact, if we choose $0 < \rho_1 < \rho_0$, then the solution $\tilde{m}$ of the problem with jump $\tilde{v}$, where $\tilde{v}$ is equal to v for $|z| < \rho_1$ and $\mathbb{I}$ otherwise, differs from m only by an exponentially small error.

This already indicates, that we should be able to replace $R_j(z)$ by their respective values at $z = 0$. To see this we start by rewriting our Riemann–Hilbert problem as a singular integral equation. We will use the theory developed in Appendix B for the case of 2×2 matrix valued functions with $m_0(z) = \mathbb{I}$ and the usual Cauchy kernel (since we won't require symmetry in this section)

$$\Omega_\infty(s, z) = \mathbb{I}\frac{ds}{s - z}.$$

Moreover, since our contour is unbounded, we will again assume $w \in L^\infty(\Sigma) \cap L^2(\Sigma)$. All results from Appendix B still hold in this case with some straightforward modifications, as the only difference is that μ is now a matrix and no longer a vector. Indeed, as in Theorem B.3, in the special case $b_+(z) = v_j(z)$ and $b_-(z) = \mathbb{I}$ for $z \in \Sigma_j$, we obtain

$$m(z) = \mathbb{I} + \frac{1}{2\pi\mathrm{i}} \int_\Sigma \mu(s) w(s) \frac{ds}{s - z}, \tag{A.9}$$

where $\mu - \mathbb{I}$ is the solution of the singular integral equation

$$(\mathbb{I} - C_w)(\mu - \mathbb{I}) = C_w \mathbb{I}, \tag{A.10}$$

that is,

$$\mu = \mathbb{I} + (\mathbb{I} - C_w)^{-1} C_w \mathbb{I}, \qquad C_w f = \mathcal{C}_-(wf). \tag{A.11}$$

Here $\mathcal{C}$ denotes the usual Cauchy operator and we set $w(z) = w_+(z)$ (since $w_-(z) = 0$).

As our first step we will get rid of some constants and rescale the entire problem by setting

$$\hat{m}(z) = D(t)^{-1} m(z t^{-1/2}) D(t), \tag{A.12}$$

where

$$D(t) = \begin{pmatrix} d(t)^{-1} & 0 \\ 0 & d(t) \end{pmatrix}, \quad d(t) = \mathrm{e}^{t\Phi(0)/2} t^{\mathrm{i}\nu/2}. \tag{A.13}$$

Then one easily checks that $\hat{m}(z)$ solves the Riemann–Hilbert problem

$$\begin{aligned} \hat{m}_+(z) &= \hat{m}_-(z) \hat{v}_j(z), && z \in \Sigma_j, \quad j = 1, 2, 3, 4, \\ \hat{m}(z) &\to \mathbb{I}, && z \to \infty, \quad z \notin \Sigma, \end{aligned} \tag{A.14}$$

where $\hat{v}_j(z) = D(t)^{-1} v_j(zt^{-1/2}) D(t)$, $j = 1, \dots, 4$, explicitly

$$\hat{v}_1(z) = \begin{pmatrix} 1 & -R_1(zt^{-1/2}) z^{2\mathrm{i}\nu} \mathrm{e}^{-t(\Phi(zt^{-1/2})-\Phi(0))} \\ 0 & 1 \end{pmatrix},$$
$$\hat{v}_2(z) = \begin{pmatrix} 1 & 0 \\ R_2(zt^{-1/2}) z^{-2\mathrm{i}\nu} \mathrm{e}^{t(\Phi(zt^{-1/2})-\Phi(0))} & 1 \end{pmatrix},$$
$$\hat{v}_3(z) = \begin{pmatrix} 1 & -R_3(zt^{-1/2}) z^{2\mathrm{i}\nu} \mathrm{e}^{-t(\Phi(zt^{-1/2})-\Phi(0))} \\ 0 & 1 \end{pmatrix},$$
$$\hat{v}_4(z) = \begin{pmatrix} 1 & 0 \\ R_2(zt^{-1/2}) z^{-2\mathrm{i}\nu} \mathrm{e}^{t(\Phi(zt^{-1/2})-\Phi(0))} & 1 \end{pmatrix}. \tag{A.15}$$

Our next aim is to show that the solution $\hat{m}(z)$ of the rescaled problem is close to the solution $\hat{m}^c(z)$ of the Riemann–Hilbert problem

$$\begin{aligned} &\hat{m}^c_+(z) = \hat{m}^c_-(z) \hat{v}^c_j(z), && z \in \Sigma_j, \quad j = 1, 2, 3, 4, \\ &\hat{m}^c(z) \to \mathbb{I}, && z \to \infty, \quad z \notin \Sigma, \end{aligned} \tag{A.16}$$

associated with the following jump matrices

$$\hat{v}^c_1(z) = \begin{pmatrix} 1 & -\overline{r} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2} \\ 0 & 1 \end{pmatrix}, \qquad \hat{v}^c_2(z) = \begin{pmatrix} 1 & 0 \\ r z^{-2\mathrm{i}\nu} \mathrm{e}^{\mathrm{i}z^2/2} & 1 \end{pmatrix},$$
$$\hat{v}^c_3(z) = \begin{pmatrix} 1 & -\frac{\overline{r}}{1-|r|^2} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2} \\ 0 & 1 \end{pmatrix}, \quad \hat{v}^c_4(z) = \begin{pmatrix} 1 & 0 \\ \frac{r}{1-|r|^2} z^{-2\mathrm{i}\nu} \mathrm{e}^{\mathrm{i}z^2/2} & 1 \end{pmatrix}. \tag{A.17}$$

The difference between these jump matrices can be estimated as follows.

Lemma A.3. *The matrices $\hat{w}^c$ and $\hat{w}$ are close in the sense that*

$$\hat{w}_j(z) = \hat{w}^c_j(z) + O(t^{-\alpha/2} \mathrm{e}^{-|z|^2/8}), \quad z \in \Sigma_j, \quad j = 1, \dots 4. \tag{A.18}$$

Furthermore, the error term is uniform with respect to parameters as stated in Theorem A.1.

Proof. We only give the proof $z \in \Sigma_1$, the other cases being similar. There is only one nonzero matrix entry in $\hat{w}_1(z) - \hat{w}^c_1(z)$ given by

$$W = \begin{cases} -R_1(zt^{-1/2}) z^{2\mathrm{i}\nu} \mathrm{e}^{-t(\Phi(zt^{-1/2})-\Phi(0))} + \overline{r} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2}, & |z| \le \rho_0 t^{1/2}, \\ \overline{r} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2} & |z| > \rho_0 t^{1/2}. \end{cases}$$

A straightforward estimate for $|z| \le \rho_0 t^{1/2}$ shows

$$\begin{aligned} |W| &= \mathrm{e}^{\nu\pi/4} |R_1(zt^{-1/2}) \mathrm{e}^{-t\hat{\Phi}(zt^{-1/2})} - \overline{r}| \mathrm{e}^{-|z|^2/2} \\ &= \mathrm{e}^{\nu\pi/4} \left| R_1(zt^{-1/2} \mathrm{e}^{-t\hat{\Phi}(zt^{-1/2})} - \overline{r} \mathrm{e}^{-t\hat{\Phi}(zt^{-1/2})} + \overline{r} \mathrm{e}^{-t\hat{\Phi}(zt^{-1/2})} - \overline{r} \right| \mathrm{e}^{-|z|^2/2} \\ &\le \mathrm{e}^{\nu\pi/4} |R_1(zt^{-1/2}) - \overline{r}| \mathrm{e}^{\mathrm{Re}(-t\hat{\Phi}(zt^{-1/2})) - |z|^2/2} \\ &\quad + \mathrm{e}^{\nu\pi/4} |\mathrm{e}^{-t\hat{\Phi}(zt^{-1/2})} - 1| \mathrm{e}^{-|z|^2/2} \\ &\le \mathrm{e}^{\nu\pi/4} |R_1(zt^{-1/2}) - \overline{r}| \mathrm{e}^{-|z|^2/4} + \mathrm{e}^{\nu\pi/4} t |\hat{\Phi}(zt^{-1/2})| \mathrm{e}^{-|z|^2/4}, \end{aligned}$$

where $\hat{\Phi}(z) = \Phi(z) - \Phi(0) - \frac{\mathrm{i}}{2} z^2 = \frac{\Phi'''(0)}{6} z^3 + \dots$. Here we have used $\frac{\mathrm{i}}{2} z^2 = \frac{1}{2}|z|^2$ for $z \in \Sigma_1$, $\mathrm{Re}(-t\hat{\Phi}(zt^{-1/2})) \le |z|^2/4$ by (A.4), and $|r| < 1$. Furthermore, by (A.5) and (A.6),

$$|W| \le \mathrm{e}^{\nu\pi/4} L t^{-\alpha/2} |z|^\alpha \mathrm{e}^{-|z|^2/4} + \mathrm{e}^{\nu\pi/4} C t^{-1/2} |z|^3 \mathrm{e}^{-|z|^2/4},$$

for $|z| \le \rho_0 t^{1/2}$. For $|z| > \rho_0 t^{1/2}$ we have

$$|W| \le \mathrm{e}^{\nu\pi/4} \mathrm{e}^{-|z|^2/2} \le \mathrm{e}^{\nu\pi/4} \mathrm{e}^{-\rho_0^2 t/4} \mathrm{e}^{-|z|^2/4}$$

which finishes the proof. □

The next lemma allows us, to replace $\hat{m}(z)$ by $\hat{m}^c(z)$.

Lemma A.4. *Consider the Riemann–Hilbert problem*

$$\begin{aligned} m_+(z) &= m_-(z) v(z), && z \in \Sigma, \\ m(z) &\to \mathbb{I}, && z \to \infty, \quad z \notin \Sigma. \end{aligned} \tag{A.19}$$

Assume that $w \in L^2(\Sigma) \cap L^\infty(\Sigma)$. Then

$$\|\mu - \mathbb{I}\|_2 \le \frac{c\|w\|_2}{1 - c\|w\|_\infty} \tag{A.20}$$

provided $c\|w\|_\infty < 1$, where c is the norm of the Cauchy operator on $L^2(\Sigma)$.

Proof. We know that $\tilde{\mu} = \mu - \mathbb{I} \in L^2(\Sigma)$ and satisfies $(\mathbb{I} - C_w)\tilde{\mu} = C_w \mathbb{I}$. Thus we can estimate $\tilde{\mu}$ by using Neumann series as follows:

$$\begin{aligned} \|\tilde{\mu}\|_2 &= \left\|(\mathbb{I} - C_w)^{-1} C_w\right\|_2 \\ &= \left\|(\mathbb{I} + C_w + C_w^2 + \dots) C_w\right\|_2 \\ &\le \|C_w\|_2 + \left\|C_w^2\right\|_2 + \left\|C_w^3\right\|_2 + \dots \\ &\le c\,\|w\|_2 + c^2\,\|w\|_2\,\|w\|_\infty + c^3\,\|w\|_2\,\|w\|_\infty^2 + \dots \\ &\le c\,\|w\|_2\,(1 + c\,\|w\|_\infty + c^2\,\|w\|_\infty^2 + \dots) \\ &= c\,\|w\|_2\,\frac{1}{1 - c\,\|w\|_\infty}. \end{aligned}$$

Here we have used that

$$\|C_w(f)\|_2 \le c\,\|f\|_2\,\|w\|_\infty\,. \tag{A.21}$$

□

Lemma A.5. *The solution $\hat{m}(z)$ has a convergent asymptotic expansion*

$$\hat{m}(z) = \mathbb{I} + \frac{1}{z}\hat{M}(t) + O(\frac{1}{z^2}) \tag{A.22}$$

for $|z| > \rho_0 t^{1/2}$ with the error term uniformly in t. Moreover,

$$\hat{M}(t) = \hat{M}^c + O(t^{-\alpha/2}). \tag{A.23}$$

Proof. Consider $\hat{m}^d(z) = \hat{m}(z)\hat{m}^c(z)^{-1}$, whose jump matrix is given by

$$\begin{aligned}\hat{v}^d(z) &= \hat{m}^c_-(z)\hat{v}(z)\hat{v}^c(z)^{-1}\hat{m}^c_-(z)^{-1}\\ &= \hat{m}^c_-(z)(\mathbb{I}+\hat{w}(z))(\mathbb{I}+\hat{w}^c(z))^{-1}\hat{m}^c_-(z)^{-1}\\ &= \hat{m}^c_-(z)(\mathbb{I}+\hat{w}(z))(\mathbb{I}-\hat{w}^c(z))\hat{m}^c_-(z)^{-1}\\ &= \hat{m}^c_-(z)(\mathbb{I}+\hat{w}(z)-\hat{w}^c(z)-\hat{w}(z)\hat{w}^c(z))\hat{m}^c_-(z)^{-1}\\ &= \mathbb{I}+\hat{m}^c_-(z)(\hat{w}(z)-\hat{w}^c(z))\hat{m}^c_-(z)^{-1}.\end{aligned}$$

By Lemma A.3, we have that $\hat{w}-\hat{w}^c$ is decaying of order $t^{-\alpha/2}$ in the norms of L^1 and L^∞ and hence also in the norm of L^2. Thus the same is true for $\hat{w}^d = \hat{v}^d - \mathbb{I} = \hat{m}^c_-(z)(\hat{w}(z)-\hat{w}^c(z))\hat{m}^c_-(z)^{-1}$. Hence by the previous lemma

$$\|\hat{\mu}^d - \mathbb{I}\|_2 = O(t^{-\alpha/2}).$$

Furthermore, by $\hat{\mu}^d = \hat{m}^d_- = \hat{m}_-(\hat{m}^c_-)^{-1} = \hat{\mu}(\hat{\mu}^c)^{-1}$ we infer

$$\|\hat{\mu}-\hat{\mu}^c\|_2 = \left\|\hat{\mu}^d\hat{\mu}^c - \hat{\mu}^c\right\|_2 = O(t^{-\alpha/2})$$

since $\hat{\mu}^c$ is bounded. Now

$$\begin{aligned}\hat{m}(z) &= \mathbb{I} + \frac{1}{2\pi\mathrm{i}}\int_\Sigma \hat{\mu}(s)\hat{w}(s)\frac{1}{s-z}ds\\ &= \mathbb{I} - \frac{1}{2\pi\mathrm{i}}\frac{1}{z}\int_\Sigma \hat{\mu}(s)\hat{w}(s)\sum_{l=0}^{\infty}\left(\frac{s}{z}\right)^l ds\\ &= \mathbb{I} - \frac{1}{2\pi\mathrm{i}}\frac{1}{z}\int_\Sigma \hat{\mu}(s)\hat{w}(s)ds + \frac{1}{2\pi\mathrm{i}}\frac{1}{z}\int_\Sigma s\hat{\mu}(s)\hat{w}(s)\frac{ds}{s-z}\end{aligned}$$

shows (recall that $\hat{w}$ is supported inside $|z| \le \rho_0 t^{1/2}$)

$$\hat{m}(z) = \mathbb{I} + \frac{1}{z}\hat{M}(t) + O(\frac{\|\hat{\mu}(s)\|_2\|s\hat{w}(s)\|_2}{z^2}),$$

where

$$\hat{M}(t) = -\frac{1}{2\pi\mathrm{i}}\int_\Sigma \hat{\mu}(s)\hat{w}(s)ds.$$

Now the rest follows from

$$\hat{M}(t) = \hat{M}^c - \frac{1}{2\pi\mathrm{i}}\int_\Sigma(\hat{\mu}(s)\hat{w}(s) - \hat{\mu}^c(s)\hat{w}^c(s))ds$$

using $\|\hat{\mu}\hat{w}-\hat{\mu}^c\hat{w}^c\|_1 \le \|\hat{w}-\hat{w}^c\|_1 + \|\hat{\mu}-\mathbb{I}\|_2\|\hat{w}-\hat{w}^c\|_2 + \|\hat{\mu}-\hat{\mu}^c\|_2\|\hat{w}^c\|_2$. □

A.2 Solving the Riemann–Hilbert problem on a small cross with constant jumps

Finally, it remains to solve (A.16) and to show:

Theorem A.6. *The solution of the Riemann–Hilbert problem (A.16) is of the form*

$$\hat{m}^c(z) = \mathbb{I} + \frac{1}{z}\hat{M}^c + O(\frac{1}{z^2}), \tag{A.24}$$

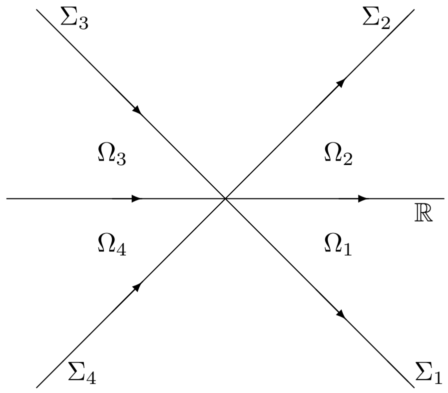

Figure A.2: Deforming back the cross

where

$$\hat{M}^c = \mathrm{i}\begin{pmatrix} 0 & -\overline{\beta} \\ \beta & 0 \end{pmatrix}, \quad \beta = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(r) + \arg(\Gamma(\mathrm{i}\nu)))}. \tag{A.25}$$

The error term is uniform with respect to r in compact subsets of $\mathbb{D}$. Moreover, the solution is bounded (again uniformly with respect to r).

Given this result, Theorem A.1 follows from Lemma A.5

$$\begin{aligned} m(z) &= D(t)\hat{m}(zt^{1/2})D(t)^{-1} = \mathbb{I} + \frac{1}{t^{1/2}z}D(t)\hat{M}(t)D(t)^{-1} + O(z^{-2}t^{-1}) \\ &= \mathbb{I} + \frac{1}{t^{1/2}z}D(t)\hat{M}^c D(t)^{-1} + O(t^{-(1+\alpha)/2}) \end{aligned} \tag{A.26}$$

for $|z| > \rho_0$, since $D(t)$ is bounded.

The proof of this result will be given in the remainder of this section. In order to solve (A.16) we begin with a deformation which moves the jump to $\mathbb{R}$ as follows. Denote the region enclosed by $\mathbb{R}$ and Σ_j as Ω_j (cf. Figure A.2) and define

$$\tilde{m}^c(z) = \hat{m}^c(z)\begin{cases} D_0(z)D_j, & z \in \Omega_j,\ j = 1, \dots, 4, \\ D_0(z), & \text{else}, \end{cases} \tag{A.27}$$

where

$$D_0(z) = \begin{pmatrix} z^{\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4} & 0 \\ 0 & z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4} \end{pmatrix},$$

and

$$D_1 = \begin{pmatrix} 1 & \overline{r} \\ 0 & 1 \end{pmatrix} \quad D_2 = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \quad D_3 = \begin{pmatrix} 1 & -\frac{\overline{r}}{1-|r|^2} \\ 0 & 1 \end{pmatrix} \quad D_4 = \begin{pmatrix} 1 & 0 \\ -\frac{r}{1-|r|^2} & 1 \end{pmatrix}.$$

Lemma A.7. *The function $\tilde{m}^c(z)$ defined in (A.27) satisfies the Riemann–Hilbert problem*

$$\begin{aligned} \tilde{m}^c_+(z) &= \tilde{m}^c_-(z)\begin{pmatrix} 1-|r|^2 & -\overline{r} \\ r & 1 \end{pmatrix}, && z \in \mathbb{R} \\ \tilde{m}^c(z) &= (\mathbb{I} + \frac{1}{z}\hat{M}^c + \dots)D_0(z), && z \to \infty,\ \frac{\pi}{4} < \arg(z) < \frac{3\pi}{4}. \end{aligned} \tag{A.28}$$

Proof. First, one checks that $\tilde{m}^c_+(z) = \tilde{m}^c_-(z)D_0(z)^{-1}\hat{v}^c_1(z)D_0(z)D_1 = \tilde{m}^c_-(z)$, $z \in \Sigma_1$ and similarly for $z \in \Sigma_2, \Sigma_3, \Sigma_4$. To compute the jump along $\mathbb{R}$ observe that, by our choice of branch cut for $z^{\mathrm{i}\nu}$, $D_0(z)$ has a jump along the negative real axis given by

$$D_{0,\pm}(z) = \begin{pmatrix} \mathrm{e}^{(\log|z|\pm \mathrm{i}\pi)\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4} & 0 \\ 0 & \mathrm{e}^{-(\log|z|\pm \mathrm{i}\pi)\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4} \end{pmatrix}, \qquad z<0.$$

Hence the jump along $\mathbb{R}$ is given by

$$D_1^{-1}D_2, \quad z>0 \quad \text{and} \quad D_4^{-1}D_{0,-}^{-1}(z)D_{0,+}(z)D_3, \quad z<0,$$

and (A.28) follows after recalling $\mathrm{e}^{-2\pi\nu} = 1-|r|^2$. □

Now, we can follow (4.17) to (4.51) in [9] to construct an approximate solution.

The idea is as follows, since the jump matrix for (A.28), the derivative $\frac{d}{dz}\tilde{m}^c(z)$ has the same jump and hence is given by $n(z)\tilde{m}^c(z)$, where the entire matrix $n(z)$ can be determined from the behavior $z\to\infty$. Since this will just serve as a motivation for our ansatz, we will not worry about justifying any steps.

For z in the sector $\frac{\pi}{4} < \arg(z) < \frac{3\pi}{4}$ (enclosed by Σ_2 and Σ_3) we have $\tilde{m}^c(z) = \hat{m}^c(z)D_0(z)$ and hence

$$\begin{aligned}
&\left(\frac{d}{dz}\tilde{m}^c(z) + \frac{\mathrm{i}z}{2}\sigma_3\tilde{m}^c(z)\right)\tilde{m}^c(z)^{-1}\\
&\quad= \left(\mathrm{i}(\frac{\nu}{z}-\frac{z}{2})\hat{m}^c(z)\sigma_3 + \frac{d}{dz}\hat{m}^c(z) + \mathrm{i}\frac{z}{2}\sigma_3\hat{m}^c(z)\right)\hat{m}^c(z)^{-1}\\
&\quad= \frac{\mathrm{i}}{2}[\sigma_3,\hat{M}^c] + O(\frac{1}{z}), \qquad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.
\end{aligned}$$

Here we assumed that the solution of the Riemann–Hilbert problem A.16 is given by (A.24) and inserted it. Since the left hand side has no jump, it is entire and hence by Liouville's theorem a constant given by the right hand side. In other words,

$$\frac{d}{dz}\tilde{m}^c(z) + \frac{\mathrm{i}z}{2}\sigma_3\tilde{m}^c(z) = \beta\tilde{m}^c(z), \quad \beta = \begin{pmatrix}0 & \beta_{12}\\ \beta_{21} & 0\end{pmatrix} = \frac{\mathrm{i}}{2}[\sigma_3,\hat{M}^c]. \qquad \text{(A.29)}$$

This differential equation can be solved in terms of parabolic cylinder function which then gives the solution of (A.28).

Lemma A.8. *The Riemann–Hilbert problem (A.28) has a unique solution, and the term $\hat{M}^c$ is given by*

$$\hat{M}^c = \mathrm{i}\begin{pmatrix}0 & -\beta_{12}\\ \beta_{21} & 0\end{pmatrix}, \qquad \beta_{12} = \overline{\beta_{21}} = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4-\arg(r)+\arg(\Gamma(\mathrm{i}\nu)))}. \qquad \text{(A.30)}$$

Proof. Uniqueness follows by the standard Liouville argument since the determinant of the jump matrix is equal to 1. We find the solution using the ansatz

$$\tilde{m}^c(z) = \begin{pmatrix}\psi_{11}(z) & \psi_{12}(z)\\ \psi_{21}(z) & \psi_{22}(z)\end{pmatrix}.$$

¿From (A.29) we can conclude that the functions $\psi_{jk}(z)$ satisfy

$$\psi_{11}''(z) = -\left(\frac{\mathrm{i}}{2} + \frac{1}{4}z^2 - \beta_{12}\beta_{21}\right)\psi_{11}(z), \qquad \psi_{12}(z) = \frac{1}{\beta_{21}}\left(\frac{d}{dz} - \frac{\mathrm{i}z}{2}\right)\psi_{22}(z),$$
$$\psi_{21}(z) = \frac{1}{\beta_{12}}\left(\frac{d}{dz} + \frac{\mathrm{i}z}{2}\right)\psi_{11}(z), \qquad \psi_{22}''(z) = \left(\frac{\mathrm{i}}{2} - \frac{1}{4}z^2 + \beta_{12}\beta_{21}\right)\psi_{22}(z).$$

That is, $\psi_{11}(\mathrm{e}^{3\pi\mathrm{i}/4}\zeta)$ satisfies the parabolic cylinder equation

$$D''(\zeta) + \left(a + \frac{1}{2} - \frac{1}{4}\zeta^2\right)D(\zeta) = 0$$

with $a = \mathrm{i}\beta_{12}\beta_{21}$ and $\psi_{22}(\mathrm{e}^{\mathrm{i}\pi/4}\zeta)$ satisfies the parabolic cylinder equation with $a = -\mathrm{i}\beta_{12}\beta_{21}$.

Let D_a be the entire parabolic cylinder function of §16.5 in [20] and set

$$\psi_{11}(z) = \begin{cases} \mathrm{e}^{-3\pi\nu/4}D_{\mathrm{i}\nu}(-\mathrm{e}^{\mathrm{i}\pi/4}z), & \mathrm{Im}(z) > 0, \\ \mathrm{e}^{\pi\nu/4}D_{\mathrm{i}\nu}(\mathrm{e}^{\mathrm{i}\pi/4}z), & \mathrm{Im}(z) < 0, \end{cases}$$
$$\psi_{22}(z) = \begin{cases} \mathrm{e}^{\pi\nu/4}D_{-\mathrm{i}\nu}(-\mathrm{i}\mathrm{e}^{\mathrm{i}\pi/4}z), & \mathrm{Im}(z) > 0, \\ \mathrm{e}^{-3\pi\nu/4}D_{-\mathrm{i}\nu}(\mathrm{i}\mathrm{e}^{\mathrm{i}\pi/4}z), & \mathrm{Im}(z) < 0. \end{cases}$$

Using the asymptotic behavior

$$D_a(z) = z^a\mathrm{e}^{-z^2/4}\big(1 - \frac{a(a-1)}{2z^2} + O(z^{-4})\big), \quad z \to \infty, \quad |\arg(z)| \le 3\pi/4,$$

shows that the choice $\beta_{12}\beta_{21} = \nu$ ensures the correct asymptotics

$$\begin{aligned} \psi_{11}(z) &= z^{\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4}(1 + O(z^{-2})), \\ \psi_{12}(z) &= -\mathrm{i}\beta_{12}z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4}(z^{-1} + O(z^{-3})), \\ \psi_{21}(z) &= \mathrm{i}\beta_{21}z^{\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4}(z^{-1} + O(z^{-3})), \\ \psi_{22}(z) &= z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4}(1 + O(z^{-2})), \end{aligned}$$

as $z \to \infty$ inside the half plane $\mathrm{Im}(z) \ge 0$. In particular,

$$\tilde{m}^c(z) = \big(\mathbb{I} + \frac{1}{z}\hat{M}^c + O(z^{-2})\big)D_0(z) \quad \text{with} \quad \hat{M}^c = \mathrm{i}\begin{pmatrix} 0 & -\beta_{12} \\ \beta_{21} & 0 \end{pmatrix}.$$

It remains to check that we have the correct jump. Since by construction both limits $\tilde{m}^c_+(z)$ and $\tilde{m}^c_-(z)$ satisfy the same differential equation (A.29), there is a constant matrix v such that $\tilde{m}^c_+(z) = \tilde{m}^c_-(z)v$. Moreover, since the coefficient matrix of the linear differential equation (A.29) has trace 0, the determinant of $\tilde{m}^c_\pm(z)$ is constant and hence $\det(\tilde{m}^c_\pm(z)) = 1$ by our asymptotics. Moreover, evaluating

$$v = \tilde{m}^c_-(0)^{-1}\tilde{m}^c_+(0) = \begin{pmatrix} \mathrm{e}^{-2\pi\nu} & -\frac{\sqrt{2\pi}\mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{-\pi\nu/2}}{\sqrt{\nu}\Gamma(\mathrm{i}\nu)}\gamma^{-1} \\ \frac{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\pi\nu/2}}{\sqrt{\nu}\Gamma(-\mathrm{i}\nu)}\gamma & 1 \end{pmatrix}$$

where $\gamma = \frac{\sqrt{\nu}}{\beta_{12}} = \frac{\beta_{21}}{\sqrt{\nu}}$. Here we have used

$$D_a(0) = \frac{2^{a/2}\sqrt{\pi}}{\Gamma((1-a)/2)}, \qquad D_a'(0) = -\frac{2^{(1+a)/2}\sqrt{\pi}}{\Gamma(-a/2)}$$

plus the duplication formula $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$ for the Gamma function. Hence, if we choose

$$\gamma = \frac{\sqrt{\nu}\Gamma(-\mathrm{i}\nu)}{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\pi\nu/2}} r,$$

we have

$$v = \begin{pmatrix} 1-|r|^2 & -\overline{r} \\ r & 1 \end{pmatrix}$$

since $|\gamma|^2 = 1$. To see this use $|\Gamma(-\mathrm{i}\nu)|^2 = \frac{\Gamma(1-\mathrm{i}\nu)\Gamma(\mathrm{i}\nu)}{-\mathrm{i}\nu} = \frac{\pi}{\nu\sinh(\pi\nu)}$ which follows from Euler's reflection formula $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$ for the Gamma function.

In particular,

$$\beta_{12} = \overline{\beta_{21}} = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(r) + \arg(\Gamma(\mathrm{i}\nu)))}$$

which finishes the proof. $\square$

Appendix B

Singular integral equations

In this chapter we show how to transform a meromorphic vector Riemann–Hilbert problem with simple poles at $\mathrm{i}\kappa$, $-\mathrm{i}\kappa$,

$$
\begin{aligned}
&m_+(k) = m_-(k)v(k), \qquad k \in \Sigma, \\
&\operatorname{Res}_{\mathrm{i}\kappa} m(k) = \lim_{k\to\mathrm{i}\kappa} m(k) \begin{pmatrix} 0 & 0 \\ \mathrm{i}\gamma^2 & 0 \end{pmatrix}, \quad \operatorname{Res}_{-\mathrm{i}\kappa} m(k) = \lim_{k\to-\mathrm{i}\kappa} m(k) \begin{pmatrix} 0 & -\mathrm{i}\gamma^2 \\ 0 & 0 \end{pmatrix}, \\
&m(-k) = m(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
&\lim_{\kappa\to\infty} m(\mathrm{i}\kappa) = \begin{pmatrix} 1 & 1 \end{pmatrix}
\end{aligned}
\tag{B.1}
$$

into a singular integral equation. Since we require the symmetry condition (1.20) for our Riemann–Hilbert problem we need to adapt the usual Cauchy kernel to preserve this symmetry. Moreover, we keep the single soliton as an inhomogeneous term which will play the role of the leading asymptotics in our applications.

B.1 Properties of the Cauchy-transform

The classical Cauchy-transform of a function $f : \Sigma \to \mathbb{C}$ which is square integrable is the analytic function $Cf : \mathbb{C}\backslash\Sigma \to \mathbb{C}$ given by

$$
Cf(k) = \frac{1}{2\pi\mathrm{i}} \int_\Sigma \frac{f(s)}{s-k} ds, \qquad k \in \mathbb{C}\backslash\Sigma. \tag{B.2}
$$

Denote the tangential boundary values from both sides (taken possibly in the L^2-sense — see e.g. [5, eq. (7.2)]) by C_+f respectively C_-f. Then it is well-known that C_+ and C_- are bounded operators $L^2(\Sigma) \to L^2(\Sigma)$, which satisfy $C_+ - C_- = \mathbb{I}$ (see e.g. [5]). Moreover, one has the Plemelj–Sokhotsky formula ([16])

$$
C_\pm = \frac{1}{2}(\mathrm{i}H \pm \mathbb{I}),
$$

where

$$
Hf(k) = \frac{1}{\pi} \fint_\Sigma \frac{f(s)}{k-s} ds, \qquad k \in \Sigma, \tag{B.3}
$$

is the Hilbert transform and $⨍$ denotes the principal value integral.

In order to respect the symmetry condition we will restrict our attention to the set $L^2_s(\Sigma)$ of square integrable functions $f : \Sigma \to \mathbb{C}^2$ such that

$$f(-k) = f(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{B.4}$$

Clearly this will only be possible if we require our jump data to be symmetric as well:

Hypothesis H. B.1. *Suppose the jump data (Σ, v) satisfy the following assumptions:*

(i) *Σ consist of a finite number of smooth oriented finite curves in $\mathbb{C}$ which intersect at most finitely many times with all intersections being transversal.*

(ii) *The distance between Σ and $\{\mathrm{i}y | y \geq y_0\}$ is positive for some $y_0 > 0$ and $\pm \mathrm{i}\kappa \notin \Sigma$.*

(iii) *Σ is invariant under $k \mapsto -k$ and is oriented such that under the mapping $k \mapsto -k$ sequences converging from the positive sided to Σ are mapped to sequences converging to the negative side.*

(iv) *The jump matrix v is invertible and can be factorized according to $v = b_-^{-1} b_+ = (\mathbb{I} - w_-)^{-1}(\mathbb{I} + w_+)$, where $w_\pm = \pm(b_\pm - \mathbb{I})$ satisfy*

$$w_\pm(-k) = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} w_\mp(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k \in \Sigma. \tag{B.5}$$

(v) *The jump matrix satisfies*

$$\begin{aligned} \|w\|_\infty &= \|w_+\|_{L^\infty(\Sigma)} + \|w_-\|_{L^\infty(\Sigma)} < \infty, \\ \|w\|_2 &= \|w_+\|_{L^2(\Sigma)} + \|w_-\|_{L^2(\Sigma)} < \infty. \end{aligned} \tag{B.6}$$

Next we introduce the Cauchy operator

$$(Cf)(k) = \frac{1}{2\pi \mathrm{i}} \int_\Sigma f(s) \Omega_\kappa(s, k) \tag{B.7}$$

acting on vector-valued functions $f : \Sigma \to \mathbb{C}^2$. Here the Cauchy kernel is given by

$$\Omega_\kappa(s, k) = \begin{pmatrix} \frac{k+\mathrm{i}\kappa}{s+\mathrm{i}\kappa} \frac{1}{s-k} & 0 \\ 0 & \frac{k-\mathrm{i}\kappa}{s-\mathrm{i}\kappa} \frac{1}{s-k} \end{pmatrix} ds = \begin{pmatrix} \frac{1}{s-k} - \frac{1}{s+\mathrm{i}\kappa} & 0 \\ 0 & \frac{1}{s-k} - \frac{1}{s-\mathrm{i}\kappa} \end{pmatrix} ds, \tag{B.8}$$

for some fixed $\mathrm{i}\kappa \notin \Sigma$. In the case $\kappa = \infty$ we set

$$\Omega_\infty(s, k) = \begin{pmatrix} \frac{1}{s-k} & 0 \\ 0 & \frac{1}{s-k} \end{pmatrix} ds. \tag{B.9}$$

and one easily checks the symmetry property:

$$\Omega_\kappa(-s, -k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Omega_\kappa(s, k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{B.10}$$

The properties of C are summarized in the next lemma.

Lemma B.2. *Assume Hypothesis B.1. The Cauchy operator C has the properties, that the boundary values $C_\pm$ are bounded operators $L^2_s(\Sigma) \to L^2_s(\Sigma)$ which satisfy*

$$C_+ - C_- = \mathbb{I} \tag{B.11}$$

and

$$(Cf)(-\mathrm{i}\kappa) = (0 \quad *), \qquad (Cf)(\mathrm{i}\kappa) = (* \quad 0). \tag{B.12}$$

Furthermore, C restricts to $L^2_s(\Sigma)$, that is

$$(Cf)(-k) = (Cf)(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k \in \mathbb{C}\backslash\Sigma \tag{B.13}$$

for $f \in L^2_s(\Sigma)$ or $L^\infty_s(\Sigma)$ and if $w_\pm$ satisfy (H.B.1) we also have

$$C_\pm(fw_\mp)(-k) = C_\mp(fw_\pm)(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k \in \Sigma. \tag{B.14}$$

Proof. Everything follows from (B.10) and the fact that C inherits all properties from the classical Cauchy operator. □

B.2 Singular integral equations in the context of Riemann–Hilbert problems

We have thus obtained a Cauchy transform with the required properties. Following Section 7 and 8 of [2], we can solve our Riemann–Hilbert problem using this Cauchy operator.

Introduce the operator $C_w : L^2_s(\Sigma) \to L^2_s(\Sigma)$ by

$$C_w f = C_+(fw_-) + C_-(fw_+), \quad f \in L^2_s(\Sigma) \tag{B.15}$$

and this operator is also well-defined for $f \in L^\infty_s(\Sigma)$ and $C_w f \in L^2_s(\Sigma)$. Furthermore recall from Lemma 1.8 that the unique solution corresponding to $v \equiv \mathbb{I}$ is given by

$$\begin{aligned} m_0(k) &= \begin{pmatrix} f(k) & f(-k) \end{pmatrix}, \\ f(k) &= \frac{1}{1 + (2\kappa)^{-1}\gamma^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa)}} \left(1 + \frac{k + \mathrm{i}\kappa}{k - \mathrm{i}\kappa} (2\kappa)^{-1}\gamma^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa)} \right). \end{aligned}$$

Observe that for $\gamma = 0$ we have $f(k) = 1$ and for $\gamma = \infty$ we have $f(k) = \frac{k+\mathrm{i}\kappa}{k-\mathrm{i}\kappa}$. In particular, $f(k)$ is uniformly bounded away from $\mathrm{i}\kappa$ for all $\gamma \in [0, \infty]$.

Then we have the next result.

Theorem B.3. *Assume Hypothesis B.1.*

Suppose m solves the Riemann–Hilbert problem (B.1). Then

$$m(k) = (1 - c_0) m_0(k) + \frac{1}{2\pi\mathrm{i}} \int_\Sigma \mu(s)(w_+(s) + w_-(s))\Omega_\kappa(s, k), \tag{B.16}$$

where

$$\mu = m_+ b_+^{-1} = m_- b_-^{-1} \quad \textit{and} \quad c_0 = \left(\frac{1}{2\pi\mathrm{i}} \int_\Sigma \mu(s)(w_+(s) + w_-(s))\Omega_\kappa(s, \infty) \right)_1.$$

Here $(m)_j$ *denotes the* j*'th component of a vector. Furthermore,* μ *solves*

$$(\mathbb{I} - C_w)(\mu(k) - (1-c_0)m_0(k)) = C_w(1-c_0)m_0(k) \tag{B.17}$$

Conversely, suppose $\tilde{\mu}$ *solves*

$$(\mathbb{I} - C_w)(\tilde{\mu}(k) - m_0(k)) = C_w m_0(k), \tag{B.18}$$

and

$$\tilde{c}_0 = \left(\frac{1}{2\pi\mathrm{i}}\int_\Sigma \tilde{\mu}(s)(w_+(s)+w_-(s))\Omega_\kappa(s,\infty)\right)_1 \neq 1,$$

then m *defined via (B.16), with* $(1-c_0) = (1+\tilde{c}_0)^{-1}$ *and* $\mu = (1+\tilde{c}_0)^{-1}\tilde{\mu}$*, solves the Riemann–Hilbert problem (B.1) and* $\mu = m_\pm b_\pm^{-1}$*.*

Proof. First of all note that by (B.14) $(\mathbb{I}-C_w)$ satisfies the symmetry condition and hence so do $m_0 + (\mathbb{I}-C_w)^{-1}C_w m_0$ and m.

So if m solves (B.1) and we set $\mu = m_\pm b_\pm^{-1}$, then m satisfies an additive jump given by

$$m_+ - m_- = \mu(w_+ + w_-),$$

as the following calculation shows

$$\begin{aligned}
&m_+ - m_- \\
&\quad = (1-c_0)(m_{0,+} - m_{0,-}) + (C_+(\mu w_+) + C_+(\mu w_-) - C_-(\mu w_+) - C_-(\mu w_-)) \\
&\quad = (C_+(\mu w_+) - C_-(\mu w_+)) + (C_+(\mu w_-) - C_-(\mu w_-)) \\
&\quad = \mu(w_+ w_-).
\end{aligned}$$

Hence, if we denote the left hand side of (B.16) by $\tilde{m}$, both functions satisfy the same additive jump. So $m - \tilde{m}$ has no jump and must thus solve (B.1) with $v \equiv \mathbb{I}$. By uniqueness (Corollary 2.3) $m - \tilde{m} = \alpha m_0$ for some $\alpha \in \mathbb{C}$ and by looking at the first component at $k \to \infty$ we see $\alpha = 0$, that is $m = \tilde{m}$.

Moreover, if m is given by (B.16), then (B.11) implies

$$\begin{aligned}
m_\pm &= (1-c_0)m_0 + C_\pm(\mu w_-) + C_\pm(\mu w_+) \\
&= (1-c_0)m_0 + C_\pm(\mu w_-) + C_\mp(\mu w_+) + C_\pm(\mu w_+) - C_\mp(\mu w_+) \\
&= (1-c_0)m_0 + C_w(\mu) \pm \mu w_\pm \\
&= (1-c_0)m_0 - (\mathbb{I} - C_w)\mu + \mu(\mathbb{I} \pm w_\pm) \\
&= (1-c_0)m_0 - (\mathbb{I} - C_w)\mu + \mu b_\pm.
\end{aligned} \tag{B.19}$$

¿From this we conclude that $\mu = m_\pm b_\pm^{-1}$ solves (B.17).

Conversely, if $\tilde{\mu}$ solves (B.18), then set

$$\tilde{m}(k) = m_0(k) + \frac{1}{2\pi\mathrm{i}}\int_\Sigma \tilde{\mu}(s)(w_+(s)+w_-(s))\Omega_\zeta(s,k),$$

and the same calculation as in (B.19) implies $\tilde{m}_\pm = \tilde{\mu} b_\pm$, which implies that $m = (1+\tilde{c}_0)^{-1}\tilde{m}$ solves the Riemann-Hilbert problem (B.1). ☐

Remark B.4. *In our case* $m_0(k) \in L^\infty(\Sigma)$*, but* $m_0(k)$ *is not square integrable and so* $\mu \in L^2(\Sigma) + L^\infty(\Sigma)$ *in general.*

In the case where the contour Σ *is bounded* $m_0(k) \in L^\infty(\Sigma)$ *implies that* $m_0(k)$ *square integrable and we can directly apply* $(\mathbb{I} - C_w)^{-1}$ *to* $m_0(k)$*.*

Note also that in the special case $\gamma = 0$ we have $m_0(k) = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and we can choose κ as we please, say $\kappa = \infty$ such that $c_0 = \tilde{c}_0 = 0$ in the above theorem.

Hence we have a formula for the solution of our Riemann–Hilbert problem $m(k)$ in terms of $m_0 + (\mathbb{I} - C_w)^{-1} C_w m_0$ and this clearly raises the question of bounded invertibility of $\mathbb{I} - C_w$ as a map from $L^2_s(\Sigma) \to L^2_s(\Sigma)$. This follows from Fredholm theory (cf. e.g. [21]):

Lemma B.5. *Assume Hypothesis B.1. The operator $\mathbb{I} - C_w$ is Fredholm of index zero,*

$$\operatorname{ind}(\mathbb{I} - C_w) = 0. \tag{B.20}$$

Proof. Since one can easily check

$$(\mathbb{I} - C_w)(\mathbb{I} - C_{-w}) = (\mathbb{I} - C_{-w})(\mathbb{I} - C_w) = \mathbb{I} - T_w, \tag{B.21}$$

where

$$T_w = T_{++} + T_{+-} + T_{-+} + T_{--}, \qquad T_{\sigma_1\sigma_2}(f) = C_{\sigma_1}[C_{\sigma_2}(f w_{-\sigma_2}) w_{-\sigma_1}],$$

it suffices to check that the operators $T_{\sigma_1\sigma_2}$ are compact ([17, Thm. 1.4.3]). By Mergelyan's theorem we can approximate $w_\pm$ by rational functions and, since the norm limit of compact operators is compact, we can assume without loss that $w_\pm$ have an analytic extension to a neighborhood of Σ.

Indeed, suppose $f_n \in L^2(\Sigma)$ converges weakly to zero. Without loss we can assume f_n to be continuous. We will show that $\|T_w f_n\|_{L^2} \to 0$.

Using the analyticity of w in a neighborhood of Σ and the definition of $C_\pm$, we can slightly deform the contour Σ to some contour $\Sigma_\pm$ close to Σ, on the left, and have, by Cauchy's theorem,

$$T_{++} f_n(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Sigma_+} (C(f_n w_-)(s) w_-(s)) \Omega_\kappa(s, z).$$

Now $(C(f_n w_-) w_-)(z) \to 0$ as $n \to \infty$. Also

$$|(C(f_n w_-) w_-)(z)| < \mathit{const}\, \|f_n\|_{L^2} \|w_-\|_{L^\infty} < \mathit{const}$$

and thus, by the dominated convergence theorem, $\|T_{++} f_n\|_{L^2} \to 0$ as desired.

Moreover, considering $\mathbb{I} - \varepsilon C_w = \mathbb{I} - C_{\varepsilon w}$ for $0 \le \varepsilon \le 1$ we obtain $\operatorname{ind}(\mathbb{I} - C_w) = \operatorname{ind}(\mathbb{I}) = 0$ from homotopy invariance of the index. □

By the Fredholm alternative, it follows that to show the bounded invertibility of $\mathbb{I} - C_w$ we only need to show that $\ker(\mathbb{I} - C_w) = 0$. The latter being equivalent to unique solvability of the corresponding vanishing Riemann–Hilbert problem.

Corollary B.6. *Assume Hypothesis B.1. A unique solution of the Riemann–Hilbert problem (B.1) exists if and only if the corresponding vanishing Riemann–Hilbert problem, where the normalization condition is replaced by $m(k) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ as $k \to \infty$, has at most one solution.*

Proof. Suppose $(\mathbb{I} - C_w)\mu = 0$ for some $\mu \in L^2_s(\Sigma)$. Set $\tilde{m}(k) = (C_w)(\mu)(k)$ for $k \in \mathbb{C}\backslash\Sigma$. Then $\tilde{m}(k)$ solves the Riemann-Hilbert problem where the normalization condition is given by $\widetilde{m(k)} = \begin{pmatrix} 0 & 0 \end{pmatrix}$ as $k \to \infty$. Hence $m_\gamma(k) = m(k) + \gamma \tilde{m}(k)$ is a solution of the Riemann-Hilbert problem (B.1) for any γ. Thus uniqueness of the solution implies that $\tilde{m}(k) \equiv 0 \in \mathbb{C}\backslash\Sigma$.

The other direction follows immediately, by the fact that a solution of the vanishing Riemann-Hilbert problem is given by $\tilde{m}(k) \equiv 0$ and hence our original Riemann-Hilbert problem has a unique solution. $\square$

We are interested in comparing two Riemann–Hilbert problems associated with respective jumps w_0 and w with $\|w - w_0\|_\infty$ and $\|w - w_0\|_2$ small, where

$$\|w\|_\infty = \|w_+\|_{L^\infty(\Sigma)} + \|w_-\|_{L^\infty(\Sigma)}. \tag{B.22}$$

and

$$\|w\|_2 = \|w_+\|_{L^2(\Sigma)} + \|w_-\|_{L^2(\Sigma)} \tag{B.23}$$

For such a situation we have the following result:

Theorem B.7. *Fix a contour Σ and choose κ, $\gamma = \gamma^t$, v^t depending on some parameter $t \in \mathbb{R}$ such that Hypothesis 2.1 holds.*

Assume that w^t satisfies

$$\|w^t\|_\infty \le \rho(t) \text{ and } \|w^t\|_2 \le \rho(t) \tag{B.24}$$

for some function $\rho(t) \to 0$ as $t \to \infty$. Then $(\mathbb{I} - C_{w^t})^{-1} : L^2_s(\Sigma) \to L^2_s(\Sigma)$ exists for sufficiently large t and the solution $m(k)$ of the Riemann–Hilbert problem (B.1) *differs from the one-soliton solution $m_0^t(k)$ only by $O(\rho(t))$, where the error term depends on the distance of k to $\Sigma \cup \{\pm \mathrm{i}\kappa\}$.*

Proof. By the boundedness of the Cauchy transform we conclude that

$$\|C_{w^t}\|_{L^2_s \to L^2_s} \le const\|w^t\|_\infty \quad \text{respectively} \quad \|C_{w^t}\|_{L^\infty_s \to L^2_s} \le const\|w^t\|_2. \tag{B.25}$$

Thus by the second resolvent identity, we infer that $(\mathbb{I} - C_{w^t})^{-1}$ exists for large t and

$$\left\|(\mathbb{I} - C_{w^t})^{-1} - \right\|_{L^2_s \to L^2_s} = O(\alpha(t)).$$

Next we observe that

$$\tilde{\mu}^t - m_0^t = (\mathbb{I} - C_{w^t})^{-1} C_{w^t} m_0^t \in L^2_s$$

and we can therefore conclude

$$\begin{aligned}\|\tilde{mu}^t - m_0^t\|_{L^2_s} = \|(\mathbb{I} - C_{w^t})^{-1} C_{w^t} m_0^t\|_{L^2_s(\Sigma)} \\ const\|C_{w^t} m_0^t\|_{L^2_s} = O(\rho(t)),\end{aligned}$$

because $\|m_0^t|_{L^\infty}$ (note also $\tilde{\mu}^t = \mu_0^t = m_0^t$). Thus we have $\tilde{c}_0^t = O(\rho(t))$. Consequently $c_0^t = O(\rho(t))$ and by using the representation for $m^t(k)$ from Lemma B.3, we finally obtain $m^t(k) - m_0^t(k) = O(\rho(t))$ uniformly in k as long as it stays a positive distance away from $\Sigma \cup \{\pm \mathrm{i}\kappa\}$. $\square$

Bibliography

[1] M. J. Ablowitz and H. Segur, *Asymptotic solutions of the Korteweg-de Vries equation*, Stud. Appl. Math **57**, 13–44 (1977).

[2] R. Beals and R. Coifman, *Scattering and inverse scattering for first order systems*, Comm. in Pure and Applied Math. **37**, 39–90 (1984).

[3] R. Beals, P. Deift, and C. Tomei, *Direct and Inverse Scattering on the Real Line*, Math. Surv. and Mon. **28**, Amer. Math. Soc., Rhode Island, 1988.

[4] V. S. Buslaev and V. V. Sukhanov, *Asymptotic behavior of solutions of the Korteweg-de Vries equation*, Jour. Sov. Math. **34**, 1905–1920 (1986).

[5] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach*, Courant Lecture Notes **3**, Amer. Math. Soc., Rhode Island, 1998.

[6] P. Deift, S. Kamvissis, T. Kriecherbauer, and X. Zhou, *The Toda rarefaction problem*, Comm. Pure Appl. Math. **49**, no. 1, 35–83 (1996).

[7] P. Deift, E. Trubowitz, *Inverse scattering on the line*, Commun. Pure Appl. Math. **32**, 121–251 (1979).

[8] P. Deift, S. Venakides and X. Zhou, *The collisionless shock region for the long-time behavior of solutions of the KdV equation*, Comm. in Pure and Applied Math. **47**, 199–206 (1994).

[9] P. Deift and X. Zhou, *A steepest descent method for oscillatory Riemann–Hilbert problems*, Ann. of Math. (2) **137**, 295–368 (1993).

[10] P. Deift and X. Zhou, *Long-time Asymptotics for Integrable Systems. Higher Order Theory*, Commun. Math. Phys. **165**, 175–191 (1994).

[11] W. Eckhaus and P. Schuur, *The emergence of solitons of the Korteweg-de Vries equation from arbitrary initial conditions*, Math. Meth. in the Appl. Sci. **5**, 97–116 (1983).

[12] F. Gesztesy and H. Holden, *Trace formulas and conservation laws for nonlinear evolution equations*, Rev. Math. Phys. **6**, 51–95 (1994).

[13] H. Krüger and G. Teschl, *Long-time asymptotics of the Toda lattice for decaying initial data revisited*, arXiv:0804.4693.

[14] H. Krüger and G. Teschl, *Long-time asymptotics for the Toda lattice in the soliton region*, arXiv:0711.2793.

[15] V. A. Marchenko, *Sturm–Liouville Operators and Applications*, Birkhäuser, Basel, 1986.

[16] N. I. Muskhelishvili, *Singular Integral Equations*, P. Noordhoff Ltd., Groningen, 1953.

[17] S. Prössdorf, *Some Classes of Singular Equations*, North-Holland, Amsterdam, 1978.

[18] P. Schuur, *Asymptotic Analysis of Soliton problems; An Inverse Scattering Approach*, Lecture Notes in Mathematics **1232**, Springer, 1986.

[19] H. Segur and M. J. Ablowitz, *Asymptotic solutions of nonlinear evolution equations and a Painléve transcendent*, Phys. D **3**, 165–184 (1981).

[20] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge, 1927.

[21] X. Zhou, *The Riemann–Hilbert problem and inverse scattering*, SIAM J. Math. Anal. **20-4**, 966–986 (1989).

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