Errata

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Mathematical Methods in Quantum Mechanics

With Applications to Schrödinger Operators

Graduate Studies in Mathematics, Vol. 99
American Mathematical Society, Providence, Rhode Island, 2009

The official web page of the book:
`http://www.mat.univie.ac.at/~gerald/ftp/book-schroe/`

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Updated as of June 23, 2018

Errata

Changes appear in yellow. Line $k+$ (resp., line $k-$) denotes the kth line from the top (resp., the bottom) of a page. My thanks go to the following individuals who have contributed to this list: Robert Stadler, Serge Richard, David Wimmesberger, Jason Jo, Helge Krüger, Katrin Grunert, Oliver Leingang, Gerhard Tulzer, Søren Fournais, Stephan Bogendörfer, Fernando Torres-Torija, Bob Sims, Gerardo González Robert, Oliver Skocek, Alexander Beigl, Erik Makino Bakken, Hendrik Vogt, Semra Demirel-Frank, Simon Becker, Mateusz Piorkowski, Laura Shou, Tom Koornwinder, Noema Nicolussi, Marcel Griesemer, Michael Hofacker.

Page 6. sentence before Lemma 0.4: Note that a metric space X is seprable ...

Page 9. Lemma 0.8 (v): If X is a metric space, then a compact set is also sequentially compact.

Page 10. Lemma 0.12: zero on C_2 and one on C_1.

Page 25. Proof of Lemma 0.27: $A = \{x | f(x) \neq 0\} = \bigcup_n A_n$

Page 31. Proof of Lemma 0.39: Unnumbered equation at the end:

$$\int_K |f| dx = \int_K f \, \mathrm{sign}(f)^* dx = \lim_{n\to\infty} \int f \varphi_n dx = 0,$$

Page 50. Equation (1.49) should read:

$$\operatorname*{s-lim}_{n\to\infty} A_n = A \quad :\Leftrightarrow \quad A_n\psi \to A\psi \quad \forall\psi \in \mathfrak{D}(A) \subseteq \mathfrak{D}(A_n). \tag{1.49}$$

Page 51. Proof of Lemma 1.14: (ii) follows as in Lemma 1.12 (ii).

Page 56. line 2−: Delete $(|\alpha_1|^2 + |\alpha_2|^2 = 1)$

Page 64. Second example: $f(2\pi) = \lim_{n\to\infty} \int_0^{2\pi} f_n'(t)dt = 0$

Page 64. Third example: In particular, if $A(x)$ is real-valued, then A_0 is essentially self-adjoint

Page 52. Last sentence in the proof of Theorem 1.16: Since $f - \varepsilon < f_{z_l}$ for all z_l we have $f - \varepsilon < f_\varepsilon$ and we have found a required function.

Page 64. Last sentence of the third example: $\mathfrak{D}(A) \subseteq \mathfrak{D}(\overline{A_0})$

Page 66. Last sentence of Lemma 2.7: we can also admit $z \in (0, \infty)$.

Page 67. First unnumbered equation:

$$|\ell_\varphi(\psi)| = |\langle\varphi, A\psi\rangle| \leq \|A\psi\|.$$

Page 68.

$$q_A(f) = \int_{\mathbb{R}^d} A(x)|f(x)|^2 d\mu(x) \tag{2.49}$$

Page 72.

$$\begin{aligned} \mathfrak{D}(A) &= \{\psi \in \mathfrak{H}_q | \exists\tilde{\psi} \in \mathfrak{H} : s(\varphi, \psi) = \langle\varphi, \tilde{\psi}\rangle, \forall\varphi \in \mathfrak{H}_q\}, \\ A\psi &= \tilde{\psi}. \end{aligned} \tag{2.60}$$

Page 73. Problem 2.13: Suppose a densely defined operator A_0 can be written as $A_0 = S^*S$ where S is a closable operator with $\mathfrak{D}(S) = \mathfrak{D}(A_0)$. Show that the Friedrichs extension is given by $A = S^*\overline{S}$.

Page 76. Proof of Lemma 2.16: $\|R_A(z_n)\varphi_n\|\|\varphi_n\|^{-1} \to \infty$

Page 77. Theorem 2.18: Moreover, for self-adjoint A we have $\|R_A(z)\| \leq |\operatorname{Im}(z)|^{-1}$ ...

Page 77. Proof of Theorem 2.18: The implication $\sigma(A) \subset [E, \infty) \Rightarrow (A-E) \geq 0$ is missing. It is easy to prove using the spectral theorem (and it is not needed before). However, here is a direct proof pointed out to me by Søren Fournais:

If $\sigma(A) \subseteq [E, \infty)$ we show $\langle\psi, (A - E)\psi\rangle \geq 0$ for every $\psi \in \mathfrak{D}(A)$ or, since $E - \lambda \in \rho(A)$ for every $\lambda > 0$, that $f(\lambda) = \langle\varphi, (A - E + \lambda)^{-1}\varphi\rangle > 0$ for every $\lambda > 0$ and $\|\varphi\| = 1$. By the first resolvent identity and Cauchy–Schwarz we see

$$f'(\lambda) = -\|(A + \lambda)^{-1}\varphi\|^2 \leq -f(\lambda)^2$$

and in particular f is a decreasing function. Suppose $f(\lambda_0) < 0$ for some $\lambda_0 > 0$. Then $f(\lambda_0) < 0$ for $\lambda > \lambda_0$ and integrating $f'/f^2 \le -1$ from λ_0 to λ shows

$$f(\lambda) \le \frac{f(\lambda_0)}{1 + f(\lambda_0)(\lambda - \lambda_0)}.$$

Hence $f(\lambda) \to -\infty$ as $\lambda \to \lambda_0 - f(\lambda_0)^{-1}$ contradicting the fact that f must be bounded for all λ.

Page 78. Problem 2.17: Then so does $A + B$ if $\|B\| < \|A^{-1}\|^{-1}$.

Page 81. Proof of Theorem 2.25:

$$1 \pm V = ((A + \mathrm{i}) \pm (A - \mathrm{i}))(A + \mathrm{i})^{-1} = \begin{cases} 2A(A + \mathrm{i})^{-1}, \\ 2\mathrm{i}(A + \mathrm{i})^{-1}, \end{cases}$$

Page 95. Two lines after (3.42): $|F_\psi(z)| \le \frac{M}{|\operatorname{Im}(z)|}$ and similarly

$$F_\psi(z^*) = F_\psi(z)^* \quad \text{and} \quad |F_\psi(z)| \le \frac{\|\psi\|^2}{|\operatorname{Im}(z)|} \tag{3.44}$$

Page 96. Line before (3.48): $0 \le \langle \psi, P_A(\Omega)\psi\rangle \le \|\psi\|^2$

Page 96. Proof of Lemma 3.6: To see $P_A(\mathbb{R}) = \mathbb{I}$, let $\psi \in \operatorname{Ker}(P_A(\mathbb{R}))$. Then $0 = d\mu_{\varphi,P_A(\mathbb{R})\psi}(\lambda) = \chi_{\mathbb{R}}(\lambda) d\mu_{\varphi,\psi}(\lambda) = d\mu_{\varphi,\psi}(\lambda)$ implies $\langle \varphi, R_A(z)\psi\rangle = 0$ which implies $\psi = 0$.

Page 98. Problem 3.5: $\|R_A(z)\| = \operatorname{dist}(z, \sigma(A))^{-1}$.

Page 109:

$$(\underline{D}\mu)(\lambda) \le \liminf_{\varepsilon\downarrow 0} \frac{1}{\pi} \operatorname{Im}\big(F(\lambda + \mathrm{i}\varepsilon)\big) \le \limsup_{\varepsilon\downarrow 0} \frac{1}{\pi} \operatorname{Im}\big(F(\lambda + \mathrm{i}\varepsilon)\big) \le (\overline{D}\mu)(\lambda). \tag{3.90}$$

Page 109: Proof of Theorem 3.22, line 8−: $\{(s,t)|\lambda - s < t < \lambda + s,\ 0 < s < \delta\} = \{(s,t)|\lambda - \delta < t < \lambda + \delta,\ |t - \lambda| < s < \delta\}$

Page 110: Proof of Corollary 3.25: The function $\sqrt{F(z)}$ used in the proof does not satisfy the growth condition (3.86) from Theorem 2.20. One can use the Herglotz function $\mathrm{i}\frac{\sqrt{F(z)}}{\sqrt{z+k}}$, with $k \in \mathbb{R}$, instead.

Page 113. Lemma 4.1: Suppose $f : I \times \mathbb{R} \to \mathbb{C}$ is a bounded uniformly continuous function

Page 115. Problem 4.11:

$$\chi_\Omega(A) = -\frac{1}{2\pi\mathrm{i}} \int_\Gamma R_A(z) dz,$$

Page 122. Theorem 4.13: $\sigma(A) \cap (\lambda_1, \lambda_2) = \{E\}$

Page 127. Line 15+: then the Cesàro mean of $|\langle \varphi, U(t)\psi\rangle|^2$ tends to zero

Page 127. Line 8−: due to Ruelle, Amrein, Georgescu, and Enß.

Page 129. Delete all squares in the last equation in the proof of Theorem 5.7.

Page 134. Lemma 6.2: The condition $\rho(A) \neq \emptyset$ should be added.

Page 135. Lemma 6.5: The condition that $A + B$ is closed should be added.

Page 148. Proof of Lemma 6.22:

$$KR_A(\lambda\mathrm{i}) = (KR_A(\mathrm{i}))((A - \mathrm{i})R_A(\lambda\mathrm{i}))$$

Page 150. Last line of the KLMN theorem: Explicitly we have $q_A + q \geq \gamma - b$

Page 150. 2nd line in Theorem 6.25: $\mathfrak{Q}(q) \supseteq \mathfrak{Q}(A)$.

Page 151. Equation (6.42) should read:

$$R_{q_A+q}(-\lambda) = R_A(-\lambda)^{1/2}(1 + C_q(\lambda))^{-1}R_A(-\lambda)^{1/2}. \qquad (6.42)$$

Similarly, $(1 - C)^{-1}$ has to be changed to $(1 + C)^{-1}$ in line 7 and 8 and $A_1 - \lambda$ to $A_1 + \lambda$ in line 8 of the following proof for Theorem 6.25.

Page 151. Line before equation (6.43): Furthermore, we can define $C_q(\lambda)$ for all $-z \in \rho(A)$, using

Page 155. Corollary 6.32: Then this holds for all z in the interior of Γ.

Page 156. Lemma 6.37: Suppose $\text{w-lim}_{n\to\infty} R_{A_n}(z) = R_A(z)$ for some $z \in \Gamma\backslash\mathbb{R}$.

Page 162. Last line in the proof of Lemma 7.1: Replace ∂^α by ∂_α two times.

Page 163. Third equation in the proof of Theorem 7.4:

$$= \lim_{\varepsilon\to 0} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi_\varepsilon(p) \mathrm{e}^{\mathrm{i}px} f(y) \mathrm{e}^{-\mathrm{i}py} d^n y d^n p,$$

Page 169. First line of equation (7.32):

$$\mathrm{e}^{-\mathrm{i}tH_0}\psi(x) = \frac{\mathrm{e}^{\mathrm{i}\frac{x^2}{4t}}}{(4\pi\mathrm{i}t)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\frac{y^2}{4t}} \psi(y) \mathrm{e}^{-\mathrm{i}\frac{xy}{2t}} d^n y$$

Page 178. Theorem 8.4: There exists an orthonormal basis of simultaneous eigenvectors for the operators L^2 and L_3.

Page 182. Equation (9.4) should read:

$$\int_c^d g^*(\tau f)\, r dy = W_c(g^*, f) - W_d(g^*, f) + \int_c^d (\tau g)^* f\, r dy, \qquad (9.4)$$

Equation (9.7) shoud read

$$\langle g, \tau f\rangle = W_a(g^*, f) - W_b(g^*, f) + \langle \tau g, f\rangle, \qquad f, g \in \mathfrak{D}(\tau). \tag{9.7}$$

Page 186. In the first equation of Problem 9.3: $\mathfrak{D}(A_\pm) = \{f \in L^2(I) | f \in AC_{loc}(I),\ \pm f' + \phi f \in L^2(I)\}$

Page 186. Problem 9.4: Add the assumption that a is regular. Otherwise one can also start the integration also at an arbitrary point in (a, b). Moreover, in the definition of c the square root is missing; it should read $c = \int_a^b \sqrt{\frac{r(t)}{p(t)}} dt$.

Page 187. Last paragraph in the proof of Lemma 9.4: Then we have $\mathfrak{D} \subseteq \mathfrak{D}(A_0^{**}) = \mathfrak{D}(\overline{A}_0)$ by (9.7).

Page 190. After (9.23): where $\tan(\alpha) = +\frac{BC_a^1(v)}{BC_a^2(v)}$

Page 190. Lemma 9.7: Then there exists a solution $u_a(z, x)$ of $(\tau - z)u = 0$ which is in $L^2((a, c), r\,dx)$

Page 192. Theorem 9.10: Delete "(which are simple)". And the following claim about simplicity of eigenvalues only applies to seperated boundary conditions as in Theorem 9.6.

Page 200. Equation (9.48) should read:

$$S_E(\lambda) = \begin{cases} \|s(E)\|^2, & \lambda = E, \\ 0, & \lambda \neq E. \end{cases} \tag{9.48}$$

Page 207. Second paragraph in Lemma 9.22: Let $c \in (a, b)$.

Page 209. Equation (9.88) should read:

$$\varepsilon = (2\|s(\lambda)\|_{(a,x)} \|c(\lambda)\|_{(a,x)})^{-1} \tag{9.88}$$

Page 215. Equation (9.111) should read:

$$\sigma_{ac}(A) = \overline{N_a(\tau) \cup N_b(\tau)}^{ess}. \tag{9.111}$$

Page 215. Equation (9.118) should read:

$$\int_n^{n+1} |q(x)| dx \le \left(\int_n^{n+1} |q(x)|^2 dx \right)^{1/2}. \tag{9.118}$$

Page 219. Problem 9.19: Change the hint according to:
(Hint: Let $\varphi_\varepsilon(x) = \exp(-\varepsilon^2 x^2)$ and investigate $\langle \varphi_\varepsilon, H\varphi_\varepsilon\rangle$.)

Page 223. Equation (10.14) should read:

$$\sigma_p(H^{(1)}) = \sigma_d(H^{(1)}) = \{E_j\}_{j\in\mathbb{N}_0}, \qquad E_0 < E_j < E_{j+1} < 0, \tag{10.14}$$

Page 225.

$$A\Phi = \tau\Phi, \quad \mathfrak{D}(A) = \{\Phi \in L^2(0,2\pi) | \ \Phi \in AC^1[0,2\pi], \Phi'' \in L^2(0,2\pi), \Phi(0) = \Phi(2\pi), \Phi'(0) = \Phi'(2\pi)\}. \tag{10.23}$$

Page 229. Equation (10.44) should read:

$$L^\infty_\infty((0,\infty)) \cap AC([0,\infty)) + L^2((0,\infty), r^2 dr) \tag{10.44}$$

Page 231. Equation (10.62) should read:

$$\int_0^\infty L_i^{(k)}(r) L_j^{(k)}(r) r^k \mathrm{e}^{-r} dr = \begin{cases} \frac{(j+k)!}{j!}, & i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{10.62}$$

Page 236. Theorem 10.11, 3rd line of proof: $\psi_\pm = \frac{|\psi| \pm \psi}{2}$
Moreover, the argument that the eigenvalue is simple is missing: If there were a second eigenfunction it could also be chosen positive by the above argument and hence could not be orthogonal to the first.

Page 236. Theorem 10.12, 2nd line of proof should read: is bounded and e^{-tH_n}, where $H_n = H_0 + V_n$, is positivity preserving

Page 253. The proof of Lemma 12.7 should read:
Pick $\phi \in C^\infty(\mathbb{R}^n, [0,1])$ such that $\phi(x) = 0$ for $0 \le |x| \le 1/2$ and $\phi(x) = 1$ for $1 \le |x|$. ...

$$\begin{aligned} [R_{H_0}(z), \phi_r] &= -R_{H_0}(z)[H_0(z), \phi_r] R_{H_0}(z) \\ &= -R_{H_0}(z)(\Delta\phi_r + 2(\partial\phi_r)\partial) R_{H_0}(z) \end{aligned}$$

and $\Delta\phi_r = \phi_{r/2}\Delta\phi_r$, $\|\Delta\phi_r\|_\infty \le \|\Delta\phi\|_\infty / r^2$, respectively, $(\partial\phi_r) = \phi_{r/2}(\partial\phi_r)$, $\|\partial\phi_r\|_\infty \le \|\partial\phi\|_\infty / r$,

Page 255. In the proof of Lemma 12.10 it should read:
Furthermore, using the intertwining property, we see that

$$\begin{aligned} (\Omega_\pm - \mathbb{I}) f(H_0) P_\pm = & R_H(z)(\Omega_\pm - \mathbb{I}) \tilde{f}(H_0) P_\pm \\ & + (R_H(z) - R_{H_0}(z)) \tilde{f}(H_0) P_\pm \end{aligned}$$

is compact by Lemma 12.8, where $\tilde{f}(\lambda) = (\lambda - z) f(\lambda) \in C_c^\infty((0,\infty))$.

Page 275. Problem A.8: Moreover, $y \mapsto \frac{\partial}{\partial x} f(x,y)$ is measurable

Page 278. Lemma A.25: (normalized at a point of continuity of μ)

Page 283. 2nd paragraph: exists if $\mu(x)$ (as defined in (A.3)) is differentiable at x

Page 283. Lemma A.31 should just read: The upper and lower derivatives are measurable.