PERTURBATION AND SPECTRAL THEORY FOR SINGULAR INDEFINITE STURM–LIOUVILLE OPERATORS

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ABSTRACT. We study singular Sturm-Liouville operators of the form

$$\frac{1}{r_j}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p_j\frac{\mathrm{d}}{\mathrm{d}x}+q_j\right), \qquad j=0,1,$$

in $L^2((a,b);r_j)$ with endpoints a and b in the limit point case, where, in contrast to the usual assumptions, the weight functions r_j have different signs near a and b. In this situation the associated maximal operators become self-adjoint with respect to indefinite inner products and their spectral properties differ essentially from the Hilbert space situation. We investigate the essential spectra and accumulation properties of nonreal and real discrete eigenvalues; we emphasize that here also perturbations of the indefinite weights r_j are allowed. Special attention is paid to Kneser type results in the indefinite setting and to L^1 perturbations of periodic operators.

1. INTRODUCTION

The purpose of this paper is to develop perturbation methods and to study spectral properties of singular Sturm–Liouville operators K_0 and K_1 associated with the differential expressions

$$\ell_0 = \frac{1}{r_0}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p_0\frac{\mathrm{d}}{\mathrm{d}x}+q_0\right) \quad\text{and}\quad \ell_1 = \frac{1}{r_1}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p_1\frac{\mathrm{d}}{\mathrm{d}x}+q_1\right) \qquad (1.1)$$

on some interval (a,b), where $-\infty \le a < b \le \infty$. As usual, we impose the standard assumptions $1/p_j, q_j, r_j \in L^1_{\mathrm{loc}}(a,b)$ real, $r_j \ne 0$, $p_j > 0$ a. e., and furthermore the endpoints a and b are assumed to be singular and in the limit point case. We will be interested in so-called *indefinite* Sturm–Liouville operators, i.e., we consider sign changing weight functions r_j; more precisely, here we treat the case $r_j < 0$ in a neighbourhood of a and $r_j > 0$ in a neighbourhood of b, $j = 0, 1$. In this situation the maximal operators K_j, $j = 0, 1$, associated to ℓ_j in the weighted L^2-spaces $L^2((a,b);r_j)$ are self-adjoint with respect to the corresponding Krein space inner products

$$[f,g] = \int_a^b f(x)\overline{g(x)}r_j(x)\,\mathrm{d}x, \qquad f, g \in L^2((a,b);r_j).$$

Various aspects in the spectral theory of indefinite Sturm-Liouville operators have been studied intensively in the mathematical literature and we refer the reader to [6, 9, 10, 11, 17, 23, 24, 33, 45, 59, 60, 62] for different types of eigenvalue estimates and to [19, 20, 26, 27, 41, 42, 43, 47, 48, 49, 69] and the references therein for

a discussion of so-called critical points, similarity, and special cases as, e.g., left definite problems.

A natural and intuitive approach to the spectral theory of indefinite Sturm-Liouville operators is to reduce a part of the analysis to the *definite* case near the singular endpoints via Glazman's decomposition method and to apply perturbation techniques; this idea appears already in the fundamental paper [25] and has been further applied and developed in, e.g., [3, 8, 12, 14, 18, 44]. More precisely, fix $a < \alpha < \beta < b$ such that both weights r_j are negative on (a, α) and positive on (β, b) and view the operators K_j as finite rank perturbations in resolvent sense of the block diagonal operators

$$H_j := \begin{pmatrix} -H_{j,-} & 0 & 0 \\ 0 & K_{j,\alpha\beta} & 0 \\ 0 & 0 & H_{j,+} \end{pmatrix} \tag{1.2}$$

in the Krein spaces

$$L^2((a,b); r_j) = L^2((a,\alpha); r_j) \oplus L^2((\alpha,\beta); r_j) \oplus L^2((\beta,b); r_j). \tag{1.3}$$

Note that sign changes of the weight functions are possible only inside the finite interval $[\alpha, \beta]$. Hence $K_{j,\alpha\beta}$ are regular indefinite Sturm-Liouville operators associated to ℓ_j in the Krein spaces $L^2((\alpha, \beta); r_j)$, and $H_{j,\pm}$ are definite singular Sturm-Liouville operators associated with the differential expressions

$$\tau_j = \frac{1}{|r_j|} \left(-\frac{\mathrm{d}}{\mathrm{d}x} p_j \frac{\mathrm{d}}{\mathrm{d}x} + q_j \right), \qquad j = 0, 1, \tag{1.4}$$

in the weighted L^2-Hilbert spaces $L^2((a, \alpha); -r_j)$ and $L^2((\beta, b); r_j)$. By imposing Dirichlet boundary conditions at α and β the operators $H_{j,\pm}$ become self-adjoint in the corresponding Hilbert spaces and $K_{j,\alpha\beta}$ becomes self-adjoint in the Krein space $L^2((\alpha, \beta); r_j)$. Due to the diagonal form it is clear that the spectra of H_j coincide with the union of the spectra of the diagonal entries. It is well-known that the spectrum of $K_{j,\alpha\beta}$ is purely discrete and hence the essential spectrum of the block diagonal operators H_j in (1.2) is given by the union of the essential spectra of the Hilbert space self-adjoint operators $\pm H_{j,\pm}$.

In order to conclude spectral properties of the operators K_j from (1.2) a careful analysis of the underlying rank two perturbation (the functions in the domains of H_j satisfy Dirichlet boundary conditions at α and β) is necessary, which is particularly subtle due to the indefinite nature of the problem as self-adjoint operators in Krein spaces may have a rather arbitrary spectral structure. A priori it is not even clear if the resolvent sets of K_j are nonempty, even rank one perturbations may lead to nonreal eigenvalues accumulating towards the essential spectrum, and other spectral effects can appear.

An additional substantial difficulty when comparing the indefinite Sturm-Liouville operators K_0 and K_1 stems from the fact that the operators act in different Krein spaces (as the weight functions r_0 and r_1 are different in general) and hence $H_{0,\pm}$ and $H_{1,\pm}$ act in different Hilbert spaces; at the same time also the regular indefinite Sturm-Liouville operators $K_{0,\alpha\beta}$ and $K_{1,\alpha\beta}$ act in different Krein spaces. We emphasize that perturbation theory for Sturm-Liouville operators with different weights $r_0 \neq r_1$ has not obtained much attention and to the best of our knowledge there is only the contribution [15] for the definite case that contains (nontrivial) results on the invariance of the essential spectrum.

In this paper we use recent results from perturbation theory of definite Sturm-Liouville operators from our publications [15, 16] and abstract perturbation results for self-adjoint operators in Krein spaces from [4, 10, 11] together with the above methodology to obtain a number of spectral results for singular indefinite Sturm-Liouville operators. In Theorem 2.5 we provide conditions on the coefficients r_j, p_j, q_j such that the essential spectra of K_0 and K_1 coincide,

$$\sigma_{\mathrm{ess}}(K_0) = \sigma_{\mathrm{ess}}(K_1),$$

which can be considered as one of the main results. In Corollary 2.6 we illustrate this general result in a more explicit situation, where it is assumed that $(a, b) = \mathbb{R}$ and the coefficients admit limits at the singular endpoints $\pm\infty$.

The accumulation of nonreal eigenvalues towards certain regions of the real axis and the accumulation of discrete real eigenvalues towards the essential spectrum is investigated in Section 3. In Theorem 3.1 and Theorem 3.4 these problems are treated for K_0 in a general setting in terms of the essential spectra of the operators $H_{0,\pm}$; the concept behind is the so-called local definitizability of self-adjoint operators in Krein spaces and the stability of this property under finite rank perturbations in resolvent sense; cf. [4, 10, 38, 39, 40]. We pay special attention to the accumulation of real discrete eigenvalues in the case $(a, b) = \mathbb{R}$, where the coefficients r_j, p_j, q_j admit limits at $\pm\infty$ such that a gap arises in the essential spectra of K_j. This allows to conclude Kneser type results in the spirit of [15, 51, 52] in the indefinite setting in Theorem 3.9 and Theorem 3.12; cf. [46] and also [9, 29, 31, 35, 37, 63, 64, 67].

Another interesting situation appears in Section 4 in the periodic setting under L^1-perturbations of the periodic coefficients of K_0: The band structure of the periodic operators $\pm H_{0,\pm}$ is preserved and leads to a band structure of the perturbed periodic operators $\pm H_{1,\pm}$, and hence to a band structure of K_1; cf. Theorem 4.1. An additional finite first moment condition on the coefficient differences together with [16, Theorem 2.3] combined with the results in Section 3 allows us to prove finiteness of eigenvalues in the spectral band gaps of the perturbed periodic operator K_1 outside a certain compact region; this can be viewed as an extension of a seminal result by Rofe-Beketov from the 1960s to general indefinite Sturm-Liouville operators; cf. [22, 30, 51, 64]. We also refer the reader to [9, 28, 50, 61] for other related studies on indefinite Sturm-Liouville operators in the periodic setting.

For the convenience of the reader the paper contains a short appendix on operators in Krein spaces, where some spectral properties and perturbation results for self-adjoint operators with finitely many negative squares and locally definitizable self-adjoint operators from the mathematical literature are recalled.

Throughout the paper we shall use the notions of essential and discrete spectrum for operators that are not necessarily self-adjoint in a Hilbert space. To avoid possible confusion we recall that for a closed operator T in a Hilbert space $\lambda \in \mathbb{C}$ is a *discrete eigenvalue* if $\lambda \in \sigma_{\mathrm{p}}(T)$ is an isolated point in the spectrum $\sigma(T)$ and the corresponding Riesz projection is a finite rank operator. The *essential spectrum* $\sigma_{\mathrm{ess}}(T)$ is the complement of the discrete eigenvalues in $\sigma(T)$. We emphasize that under our assumptions the essential spectra of the Sturm-Liouville operators appearing in this paper is automatically real and remains invariant under compact perturbations in resolvent sense.

2. Essential Spectrum

Let $-\infty \leq a < b \leq \infty$ and let ℓ_j and τ_j, $j = 0, 1$, be the Sturm–Liouville expressions on (a, b) in (1.1) and (1.4), respectively, and assume that the coefficients satisfy the standard assumptions $1/p_j, q_j, r_j \in L^1_{\mathrm{loc}}(a, b)$ real, $r_j \neq 0$, and $p_j > 0$ almost everywhere. The next hypothesis on the different signs of the weight functions near a and b is central for the present paper.

Hypothesis 2.1. There exist $\alpha, \beta \in \mathbb{R}$ with $a < \alpha \leq \beta < b$ such that $r_j < 0$ on (a, α) and $r_j > 0$ on (β, b) for $j = 0, 1$.

The Hilbert spaces of measurable complex valued functions f on (a, b) such that $r_j f^2 \in L^1(a, b)$ are denoted by $L^2((a, b); |r_j|)$ and are equipped with the standard scalar products

$$(f, g) = \int_a^b f(x)\overline{g(x)}|r_j(x)| \, \mathrm{d}x, \qquad f, g \in L^2((a, b); |r_j|).$$

Besides these scalar products we shall also consider the inner products

$$[f, g] = \int_a^b f(x)\overline{g(x)}r_j(x) \, \mathrm{d}x, \qquad f, g \in L^2((a, b); |r_j|), \qquad (2.1)$$

which are both indefinite by Hypothesis 2.1. The spaces $L^2((a, b); |r_j|)$ equipped with $[\cdot, \cdot]$ become Krein spaces and will be denoted by $L^2((a, b); r_j)$. The corresponding fundamental symmetries $J = \mathrm{sgn}(r_j)$ connect the inner products via $[f, g] = (Jf, g)$. Note also that the differential expressions ℓ_j are formally symmetric with respect to the indefinite inner products $[\cdot, \cdot]$ and that $\ell_j = J\tau_j$, $j = 0, 1$. For the basic properties of indefinite inner product spaces and linear operators therein we refer to [1, 21, 32].

Next we will define various Sturm-Liouville operators associated with ℓ_j and τ_j. For our purposes the following hypothesis on the definite Sturm-Liouville expression τ_0 is appropriate.

Hypothesis 2.2. The endpoints a and b are singular and in the limit point case with respect to τ_0.

Let $j = 0, 1$ and denote the maximal domains by

$$\mathcal{D}_j(a, b) = \left\{ f \in L^2((a, b); |r_j|) : f, pf' \in \mathcal{AC}(a, b), \tau_j f \in L^2((a, b); |r_j|) \right\}, \quad (2.2)$$

where $\mathcal{AC}(a, b)$ stands for the space of absolutely continuous functions on (a, b). Note that $\tau_j f \in L^2((a, b); |r_j|)$ if and only if $\ell_j f \in L^2((a, b); |r_j|)$. The maximal operators associated to ℓ_j and τ_j are defined as

$$\begin{aligned} K_j f &= \ell_j f \qquad \mathrm{dom}\, K_j = \mathcal{D}_j(a, b), \\ L_j g &= \tau_j g \qquad \mathrm{dom}\, L_j = \mathcal{D}_j(a, b), \end{aligned} \qquad (2.3)$$

for $j = 0, 1$. Observe that Hypothesis 2.2 ensures that the definite Sturm-Liouville operator L_0 is self-adjoint in the Hilbert space $L^2((a, b); |r_0|)$ and from $K_0 = JL_0$ it follows that the indefinite Sturm-Liouville operator K_0 is self-adjoint in the Krein space $L^2((a, b); r_0)$; cf. Appendix A. Besides the natural maximal operators in (2.3) we shall also make use of Sturm-Liouville operators associated to ℓ_j and τ_j on the subintervals (a, α), (α, β), and (β, b) equipped with Dirichlet boundary conditions at the regular endpoints α and β. More precisely, for $j = 0, 1$ we define the operators

$$
\begin{aligned}
H_{j,-}f &= \tau_j f, & \operatorname{dom} H_{j,-} &= \big\{ f \in \mathcal{D}_j(a, \alpha) : f(\alpha) = 0 \big\}, \\
K_{j,\alpha\beta}g &= \ell_j g, & \operatorname{dom} K_{j,\alpha\beta} &= \big\{ g \in \mathcal{D}_j(\alpha, \beta) : g(\alpha) = g(\beta) = 0 \big\}, & (2.4) \\
H_{j,+}h &= \tau_j h, & \operatorname{dom} H_{j,+} &= \big\{ h \in \mathcal{D}_j(\beta, b) : h(\beta) = 0 \big\},
\end{aligned}
$$

where the maximal domains $\mathcal{D}_j(a, \alpha)$, $\mathcal{D}_j(\alpha, \beta)$, and $\mathcal{D}_j(\beta, b)$ are defined in the same way as $\mathcal{D}_j(a, b)$ in (2.2). It follows from Hypothesis 2.2 that the operator $H_{0,-}$ is self-adjoint in the Hilbert space $L^2((a, \alpha); |r_0|)$ and the operator $H_{0,+}$ is self-adjoint in the Hilbert space $L^2((\beta, b); |r_0|)$. The next hypothesis ensures that the operators L_0 and $H_{0,\pm}$ are semibounded from below (see, e.g. [15, Proof of Theorem 3.2]).

Hypothesis 2.3. The function q_0/r_0 is bounded near a and b.

We emphasize that the regular indefinite Sturm-Liouville operators $K_{j,\alpha\beta}$ are self-adjoint in $L^2((\alpha, \beta); r_j)$ (see, e.g., [25]), where these spaces are equipped with the inner product (2.1) restricted to (α, β).

Remark 2.4. Note that the inner product (2.1) on (α, β) can be indefinite or definite, depending on the properties of the weight functions r_j, $j = 0, 1$, on (α, β). To avoid confusion we will always view $L^2((\alpha, \beta); r_j)$ as a Krein space (which reduces to a Hilbert space in the special case $r_j > 0$ a.e. on (α, β)). Furthermore, the case $\alpha = \beta$ in Hypothesis 2.1 is understood in the sense that the regular indefinite Sturm-Liouville operators $K_{j,\alpha\beta}$ and the Krein spaces $L^2((\alpha, \beta); r_j)$ are absent in the operator and space decompositions appearing later in the text.

The next result is the first main result in this paper. We provide criteria on the coefficients of ℓ_0 and ℓ_1 such that the essential spectra of the indefinite Sturm-Liouville operators K_0 and K_1 coincide. It also turns out that the assumptions on the coefficients imply that the resolvent sets of K_0 and K_1 are both nonempty.

Theorem 2.5. *Assume Hypotheses 2.1, 2.2, and 2.3, and suppose that for each endpoint $e \in \{a, b\}$ the following conditions hold:*

$$
\lim_{x \to e} \frac{r_1(x)}{r_0(x)} = 1, \quad \lim_{x \to e} \frac{p_1(x)}{p_0(x)} = 1, \quad \lim_{x \to e} \frac{q_1(x) - q_0(x)}{r_0(x)} = 0.
$$

Then both indefinite Sturm-Liouville operators K_0 and K_1 are self-adjoint in the Krein spaces $L^2((a, b); r_0)$ and $L^2((a, b); r_1)$, respectively, the resolvent sets $\rho(K_0)$ and $\rho(K_1)$ are nonempty, and

$$
\sigma_{\mathrm{ess}}(K_0) = \sigma_{\mathrm{ess}}(K_1) \subset \mathbb{R}. \tag{2.5}
$$

Proof. In the present situation it follows from [15, Theorem 3.2] that both singular endpoints a and b are also in the limit point case with respect to τ_1 and hence the maximal definite Sturm-Liouville operator L_1 associated to τ_1 is self-adjoint in $L^2((a, b); |r_1|)$. It also follows from Hypothesis 2.3 and one more application of

[15, Theorem 3.2] that the operators $H_{j,\pm}$ are self-adjoint and semibounded in the corresponding Hilbert spaces $L^2((b,\beta);r_j)$ and $L^2((a,\alpha);-r_j)$, and that

$$\sigma_{\text{ess}}(H_{0,+}) = \sigma_{\text{ess}}(H_{1,+}) \quad \text{and} \quad \sigma_{\text{ess}}(H_{0,-}) = \sigma_{\text{ess}}(H_{1,-}). \tag{2.6}$$

The semiboundedness of $H_{0,\pm}$ and $H_{1,\pm}$ also implies the semiboundedness of L_0 and L_1, respectively. Hence it follows from [12, Theorem 4.5] that both indefinite Sturm-Liouville operators K_0 and K_1 are self-adjoint in the Krein spaces $L^2((a,b);r_0)$ and $L^2((a,b);r_1)$, respectively, and that the resolvent sets $\rho(K_0)$ and $\rho(K_1)$ are nonempty.

We now turn to the essential spectra of K_0 and K_1 and verify the remaining assertion (2.5). For this we first consider the orthogonal sums H_j, $j = 0, 1$, given by

$$H_0 = \begin{pmatrix} -H_{0,-} & 0 & 0 \\ 0 & K_{0,\alpha\beta} & 0 \\ 0 & 0 & H_{0,+} \end{pmatrix} \quad \text{and} \quad H_1 = \begin{pmatrix} -H_{1,-} & 0 & 0 \\ 0 & K_{1,\alpha\beta} & 0 \\ 0 & 0 & H_{1,+} \end{pmatrix}, \tag{2.7}$$

on their natural domains

$$\text{dom}\, H_j = \text{dom}\, H_{j,-} \times \text{dom}\, K_{j,\alpha\beta} \times \text{dom}\, H_{j,+}, \quad j = 0, 1,$$

in the Krein spaces

$$L^2((a,b);r_j) = L^2((a,\alpha);r_j) \oplus L^2((\alpha,\beta);r_j) \oplus L^2((\beta,b);r_j), \quad j = 0, 1;$$

cf. (1.2) and (1.3). Note that the operators $K_{0,\alpha\beta}$ and $K_{1,\alpha\beta}$ are both regular indefinite Sturm-Liouville operators that are self-adjoint in the Krein space $L^2((\alpha,\beta);r_0)$ and $L^2((\alpha,\beta);r_1)$, respectively, and that the spectra of these operators consist of real discrete eigenvalues accumulating to ∞ and $-\infty$. In addition, there may appear at most finitely many pairs of nonreal discrete eigenvalues which are symmetric with respect to the real line. This is a consequence of the fact that $K_{0,\alpha\beta}$ and $K_{1,\alpha\beta}$ have finitely many negative squares; we refer the reader to [25] and Theorem A.3 in Appendix A.

Therefore, we conclude that H_0 and H_1 are self-adjoint in the Krein spaces $L^2((a,b);r_0)$ and $L^2((a,b);r_1)$, respectively, and from (2.6) and (2.7) we obtain that the essential spectra of H_0 and H_1 coincide and

$$\begin{aligned}
\sigma_{\text{ess}}(H_0) &= \sigma_{\text{ess}}(H_{0,+}) \cup \sigma_{\text{ess}}(-H_{0,-}) \\
&= \sigma_{\text{ess}}(H_{1,+}) \cup \sigma_{\text{ess}}(-H_{1,-}) = \sigma_{\text{ess}}(H_1).
\end{aligned} \tag{2.8}$$

As $\rho(K_0)$ and $\rho(K_1)$ are both nonempty we see that for $\lambda \in \rho(K_0) \cap \rho(H_0)$ and $\mu \in \rho(K_1) \cap \rho(H_1)$ the resolvent differences

$$(K_0 - \lambda)^{-1} - (H_0 - \lambda)^{-1} \quad \text{and} \quad (K_1 - \mu)^{-1} - (H_1 - \mu)^{-1} \tag{2.9}$$

are rank two operators. In fact, this follows from the observation that $K_j f = H_j f$ for all $f \in \mathcal{D}_j(a,b)$ such that $f(\alpha) = f(\beta) = 0$, and it is clear that this subspace of functions is a two-dimensional restriction of the maximal domains. Hence the essential spectra of K_0 and H_0 coincide and the essential spectra of K_1 and H_1 coincide, that is,

$$\sigma_{\text{ess}}(K_0) = \sigma_{\text{ess}}(H_0) \quad \text{and} \quad \sigma_{\text{ess}}(K_1) = \sigma_{\text{ess}}(H_1). \tag{2.10}$$

This observation together with (2.8) leads to (2.5); note that the essential spectrum is real as $H_{0,\pm}$ and $H_{1,\pm}$ are self-adjoint in Hilbert spaces. $\qquad\square$

In the next corollary we use Theorem 2.5 to express the essential spectrum in a more explicit way for the case where the coefficients of ℓ_0 and ℓ_1 admit the same limits $r_{\pm\infty}, p_{\pm\infty}, q_{\pm\infty}$ at $\pm\infty$.

Corollary 2.6. *Assume that $(a, b) = \mathbb{R}$ and that the coefficients r_j, p_j, q_j, $j = 0, 1$, admit the limits*

$$r_{\pm\infty} = \lim_{x\to\pm\infty} r_0(x) = \lim_{x\to\pm\infty} r_1(x),$$

$$p_{\pm\infty} = \lim_{x\to\pm\infty} p_0(x) = \lim_{x\to\pm\infty} p_1(x),$$

$$q_{\pm\infty} = \lim_{x\to\pm\infty} q_0(x) = \lim_{x\to\pm\infty} q_1(x),$$

where $\pm r_{\pm\infty} > 0$, $p_{\pm\infty} > 0$, and $q_{\pm\infty} \in \mathbb{R}$. Then both indefinite Sturm-Liouville operators K_0 and K_1 are self-adjoint in the Krein spaces $L^2(\mathbb{R}; r_0)$ and $L^2(\mathbb{R}; r_1)$, respectively, the resolvent sets $\rho(K_0)$ and $\rho(K_1)$ are nonempty, and

$$\sigma_{\mathrm{ess}}(K_0) = \sigma_{\mathrm{ess}}(K_1) = \left(-\infty, \frac{q_{-\infty}}{r_{-\infty}}\right] \cup \left[\frac{q_{+\infty}}{r_{+\infty}}, \infty\right). \tag{2.11}$$

In particular, there is a gap in the essential spectrum if $q_{-\infty}/r_{-\infty} < q_{+\infty}/r_{+\infty}$.

Proof. We shall define a suitable comparison operator K with constant coefficients and apply Theorem 2.5 to the pairs $\{K, K_0\}$ and $\{K, K_1\}$. Consider the piecewise constant coefficients r, p, q defined by

$$r(x) = \begin{cases} r_{+\infty} & \text{if } x \geq 0, \\ r_{-\infty} & \text{if } x < 0, \end{cases} \quad p(x) = \begin{cases} p_{+\infty} & \text{if } x \geq 0, \\ p_{-\infty} & \text{if } x < 0 \end{cases}, \quad q(x) = \begin{cases} q_{+\infty} & \text{if } x \geq 0, \\ q_{-\infty} & \text{if } x < 0, \end{cases}$$

and note that the corresponding definite Sturm-Liouville expression

$$\tau = \frac{1}{|r|} \left(-\frac{\mathrm{d}}{\mathrm{d}x} p \frac{\mathrm{d}}{\mathrm{d}x} + q \right)$$

is in the limit point case at both singular endpoints $\pm\infty$, that is, Hypothesis 2.2 holds. From the choice of q and r, and the assumptions on r_0, r_1 it is also clear that Hypothesis 2.1 and Hypothesis 2.3 are satisfied. Now we associate the operators $K, L, H_+, K_{\alpha\beta}, H_-$ to τ and its indefinite counterpart $\ell = \mathrm{sgn}(r)\tau$ in the same way as in (2.2), (2.3), and (2.4). Then the essential spectrum of the operators

$$H_- f = \tau f, \qquad \mathrm{dom}\, H_- = \{f \in \mathcal{D}(-\infty, \alpha) : f(\alpha) = 0\},$$

$$H_+ h = \tau h, \qquad \mathrm{dom}\, H_+ = \{h \in \mathcal{D}(\beta, \infty) : h(\beta) = 0\},$$

is given by

$$\sigma_{\mathrm{ess}}(H_+) = \left[\frac{q_{+\infty}}{r_{+\infty}}, \infty\right) \quad \text{and} \quad \sigma_{\mathrm{ess}}(H_-) = \left[-\frac{q_{-\infty}}{r_{-\infty}}, \infty\right) \tag{2.12}$$

(cf. [65, Example on p. 209]), and hence we conclude in the same way as in (2.7), (2.8), and (2.10) that

$$\sigma_{\mathrm{ess}}(K) = \left(-\infty, \frac{q_{-\infty}}{r_{-\infty}}\right] \cup \left[\frac{q_{+\infty}}{r_{+\infty}}, \infty\right). \tag{2.13}$$

It is easy to see that the conditions

$$\lim_{x\to\pm\infty} \frac{r_j(x)}{r(x)} = 1, \qquad \lim_{x\to\pm\infty} \frac{p_j(x)}{p(x)} = 1, \qquad \lim_{x\to\pm\infty} \frac{q_j(x) - q(x)}{r(x)} = 0$$

hold for $j = 0, 1$ and hence we conclude from Theorem 2.5 that the indefinite Sturm-Liouville operators K_0 and K_1 are self-adjoint in the Krein spaces $L^2(\mathbb{R}; r_0)$ and $L^2(\mathbb{R}; r_1)$, the resolvent sets $\rho(K_0)$ and $\rho(K_1)$ are nonempty, and (2.11) follows from (2.13) and (2.5). $\hfill\square$

3. Discrete spectrum

In this section we study the point spectrum of the indefinite Sturm-Liouville operator K_0 from the previous section. Here we concentrate on the nonreal point spectrum and accumulation points in gaps of the essential spectrum. Recall from the previous section that under Hypotheses 2.1, 2.2, and 2.3 the essential spectrum of K_0 is given by

$$\sigma_{\mathrm{ess}}(K_0) = \sigma_{\mathrm{ess}}(H_{0,+}) \cup \sigma_{\mathrm{ess}}(-H_{0,-}) \tag{3.1}$$

and hence it is clear that the nonreal spectrum of K_0 consists of discrete eigenvalues that may only accumulate to points in (3.1). In the next theorem we observe that accumulation of nonreal eigenvalues is possible only towards certain subsets of (3.1).

Theorem 3.1. *Assume Hypotheses 2.1, 2.2, and 2.3. Then the following assertions hold.*

(i) *The nonreal spectrum of K_0 consists of discrete eigenvalues with geometric multiplicity one which are contained in a compact subset of $\mathbb{C}$.*

(ii) *The nonreal spectrum may only accumulate to points in*

$$\sigma_{\mathrm{ess}}(H_{0,+}) \cap \sigma_{\mathrm{ess}}(-H_{0,-}).$$

Furthermore, the nonreal spectrum cannot accumulate to any of the points λ from the boundary (in $\mathbb{R}$)

$$\partial\left(\sigma_{\mathrm{ess}}(H_{0,+}) \cap \sigma_{\mathrm{ess}}(-H_{0,-})\right) \tag{3.2}$$

with the following property (P): *There exists $\varepsilon > 0$ such that*

$$(\lambda - \varepsilon, \lambda) \subset \rho(H_{0,+}) \quad and \quad (\lambda, \lambda + \varepsilon) \subset \rho(-H_{0,-})$$

or

$$(\lambda - \varepsilon, \lambda) \subset \rho(-H_{0,-}) \quad and \quad (\lambda, \lambda + \varepsilon) \subset \rho(H_{0,+}).$$

(iii) *In particular, if the interior (in $\mathbb{R}$)*

$$\mathrm{int}\left(\sigma_{\mathrm{ess}}(H_{0,+}) \cap \sigma_{\mathrm{ess}}(-H_{0,-})\right) = \emptyset$$

and all λ in the set (3.2) *satisfy property* (P), *then K_0 has at most finitely many nonreal eigenvalues.*

Proof. We shall first show in Step 1 and Step 2 that the self-adjoint operator

$$H_0 = \begin{pmatrix} -H_{0,-} & 0 & 0 \\ 0 & K_{0,\alpha\beta} & 0 \\ 0 & 0 & H_{0,+} \end{pmatrix} \tag{3.3}$$

acting in the Krein space

$$L^2((a, b); r_0) = L^2((a, \alpha); r_0) \oplus L^2((\alpha, \beta); r_0) \oplus L^2((\beta, b); r_0);$$

is definitizable over the domain

$$\left(\overline{\mathbb{C}} \setminus (\sigma_{\mathrm{ess}}(H_{0,+}) \cap \sigma_{\mathrm{ess}}(-H_{0,-}))\right)$$
$$\cup \left\{\lambda \in \partial\left(\sigma_{\mathrm{ess}}(H_{0,+}) \cap \sigma_{\mathrm{ess}}(-H_{0,-})\right) : \lambda \text{ has property (P)}\right\} \tag{3.4}$$

in the sense of Definition A.4 (see also [38, 40] and [66]). Using the perturbation result in Theorem A.5 (i) we conclude in Step 3 that the same is true for K_0, which implies the assertions (i)-(iii).

Step 1. We verify Definition A.4 (ii). By Hypothesis 2.1 $L^2((a, \alpha); r_0)$ is an anti-Hilbert space, $L^2((\beta, b); r_0)$ is a Hilbert space, and $L^2((\alpha, \beta); r_0)$ is (in general) a Krein space. Since $H_{0,+}$ and $-H_{0,-}$ are self-adjoint in Hilbert spaces their spectra are real. In the present situation $\sigma(H_{0,+})$ is of positive type in the sense of Definition A.1 and $\sigma(-H_{0,-})$ is of negative type in the sense of Definition A.1. As mentioned in the proof of Theorem 2.5 the operator $K_{0,\alpha\beta}$ is a regular indefinite Sturm-Liouville operator with finitely many negative squares (for the notion of negative squares we refer to the Appendix) and the spectrum consists only of discrete eigenvalues; the real eigenvalues accumulate to ∞ and $-\infty$. Hence for every real λ not in $\sigma_{\mathrm{ess}}(H_{0,+}) \cap \sigma_{\mathrm{ess}}(-H_{0,-})$ and all points in (3.2) which satisfy property (P) item (ii) from Definition A.4 is satisfied for the operator H_0 in (3.3).

We show next that item (ii) from Definition A.4 is also satisfied for $\lambda = \infty$. In fact, it follows from Hypothesis 2.3 that $H_{0,+}$ is semibounded from below and $-H_{0,-}$ is semibounded from above and hence $(-\infty, -\gamma) \subset \rho(H_{0,+})$ and $(\gamma, \infty) \subset \rho(-H_{0,-})$ for some $\gamma \geq 0$. As mentioned above the spectrum of $H_{0,+}$ is of positive type and the spectrum of $-H_{0,-}$ is of negative type. Since $K_{0,\alpha\beta}$ has finitely many negative squares Theorem A.3 (iii) implies that one can choose γ in such a way that $(-\infty, -\gamma) \cap \sigma(K_{0,\alpha\beta})$ is of negative type and $(\gamma, \infty) \cap \sigma(K_{0,\alpha\beta})$ is of positive type. Now the claim follows for $\lambda = \infty$.

Step 2. Observe that the nonreal spectrum of H_0 coincides with the nonreal spectrum of $K_{0,\alpha\beta}$ and hence it follows from Theorem A.3 (i) that the nonreal spectrum of H_0 consists of isolated points which are poles of the resolvent of H_0 and no point of $\mathbb{R} \cup \{\infty\}$ is an accumulation point of non-real spectrum of H_0. It remains to verify Definition A.4 (i). In fact, the growth condition for the resolvent of H_0 is valid since the resolvents of $H_{0,+}$ and $-H_{0,-}$ are bounded by $|\mathrm{Im}\,\lambda|^{-1}$ for non-real λ and the operator $K_{0,\alpha\beta}$ satisfies the growth condition for the resolvent by Theorem A.3 (iv). Hence H_0 is definitizable over the domain (3.4).

Step 3. By (2.9) the difference of the resolvents of H_0 and K_0 is a rank two operator and hence Theorem A.5 implies that the operator K_0 is also definitizable over the domain (3.4); note that $\rho(K_0) \neq \emptyset$ follows from Hypothesis 2.3 and [12, Theorem 4.5]. This yields (ii) in Theorem 3.1. Observe that the nonreal spectrum of K_0 is discrete by (3.1) and Hypothesis 2.2 implies that all eigenvalues of K_0 have geometric multiplicity one. As ∞ is contained in the domain (3.4) it is also clear that ∞ is not an accumulation point of nonreal eigenvalues of K_0. This shows (i) and assertion (iii) is an immediate consequence of (i) and (ii). $\square$

Next we illustrate the decisive role of points with property (P) from Theorem 3.1 (ii) with some examples.

Example 3.2. Let $(a, b) = \mathbb{R}$ and consider the shifted Coulomb potential

$$r_0(x) = \mathrm{sgn}\,x, \quad p_0(x) = 1 \quad \text{and} \quad q_0(x) = -\frac{1}{1 + |x|}, \quad x \in \mathbb{R},$$

where $\alpha = \beta = 0$ in Hypothesis 2.1, see also [7, 59] and [34]. In this situation Corollary 2.6 and (2.12) show $\sigma_{\mathrm{ess}}(H_{0,+}) = [0, \infty)$ and $\sigma_{\mathrm{ess}}(-H_{0,-}) = (-\infty, 0]$ and

hence (3.2) turns into

$$\partial \left(\sigma_{\text{ess}}(H_{0,+}) \cap \sigma_{\text{ess}}(-H_{0,-}) \right) = \{0\}. \tag{3.5}$$

Note that the operators $\pm H_{0,\pm}$ from (2.4) act in $L^2(\mathbb{R}_\pm)$ and the operator $K_{0,\alpha\beta}$ is not present in the decomposition (1.2); cf. Remark 2.4. A Kneser type argument (see, e.g., [65, Corollary 9.43]) shows that the discrete spectrum in $(-\infty, 0)$ of the operator $H_{0,+}$ accumulates to zero from below and the discrete spectrum in $(0, \infty)$ of the operator $-H_{0,-}$ accumulates to zero from above. Hence property (P) is not fulfilled for the point zero and thus accumulation of nonreal eigenvalues of K_0 to zero may appear.

In fact, it was shown in [59] that for the shifted (indefinite) Coulomb potential the nonreal discrete eigenvalues indeed accumulate to zero. In general it is an open problem formulated in [5] whether the accumulation of discrete eigenvalues from below to the lower boundary of the essential spectrum of $H_{0,+}$ or the accumulation of discrete eigenvalues from above to the upper boundary of the essential spectrum of $-H_{0,-}$ leads to nonreal accumulation of discrete eigenvalues to points from (3.2) not satisfying property (P).

Example 3.3. Let again $(a, b) = \mathbb{R}$, $\alpha = \beta = 0$, and consider a potential with fast decay towards $\pm\infty$ as in [8],

$$r_0(x) = \operatorname{sgn} x \quad p_0(x) = 1 \quad \text{and} \quad q_0(x) = -\kappa(\kappa+1)\operatorname{sech}^2(x), \quad x \in \mathbb{R},$$

for some $\kappa \in \mathbb{N}$. Then again the essential spectrum of $\pm H_{0,\pm}$ coincides with $\mathbb{R}_\pm$ and (3.5) holds. In contrast to Example 3.2 here a Kneser type argument shows that the discrete spectra of $H_{0,+}$ in $(-\infty, 0)$ and of $-H_{0,-}$ in $(0, \infty)$ are finite. Hence, zero has property (P) and Theorem 3.1 (iii) shows that the operator K_0 has finitely many nonreal eigenvalues; cf. [8].

In the next theorem we complement the previous observations in Theorem 3.1 and study the possible accumulation of real discrete eigenvalues of K_0 to boundary points of (3.1). In this context we recall from [36, Corollary 6] that $\sigma_{\text{ess}}(\pm H_{0,\pm})$ can be any closed subset of $\mathbb{R}$ and, in particular, spectral points as in Theorem 3.4 (iii) below may appear.

Theorem 3.4. *Assume Hypotheses 2.1, 2.2, and 2.3. Then the following assertions hold.*

(i) *If $\lambda^+ \in \partial\sigma_{\text{ess}}(H_{0,+})$ (boundary in $\mathbb{R}$) and $\lambda^+ \notin \sigma_{\text{ess}}(-H_{0,-})$, then the real discrete eigenvalues of K_0 accumulate at λ^+ if and only if the discrete eigenvalues of $H_{0,+}$ accumulate at λ^+.*

(ii) *If $\lambda^- \in \partial\sigma_{\text{ess}}(-H_{0,-})$ (boundary in $\mathbb{R}$) and $\lambda^- \notin \sigma_{\text{ess}}(H_{0,+})$, then the real discrete eigenvalues of K_0 accumulate at λ^- if and only if the discrete eigenvalues of $-H_{0,-}$ accumulate at λ^-.*

(iii) *If λ in (3.2) satisfies (P) and $\lambda \notin \operatorname{int}(\sigma_{\text{ess}}(K_0))$ (interior in $\mathbb{R}$), then the real discrete eigenvalues of K_0 accumulate at λ.*

Proof. (i) and (ii): In the proof of Theorem 3.1 it was shown that H_0 and K_0 are definitizable over the domain (3.4) in all points $\lambda \in \mathbb{R} \cup \{\infty\}$ not in $\sigma_{\text{ess}}(H_{0,+}) \cap \sigma_{\text{ess}}(-H_{0,-})$ and in all points in (3.2) which satisfy property (P). Then the statement on the accumulation of eigenvalues follows from Theorem A.5 (ii).

(iii) Assume that λ in (3.2) satisfies (P) and $\lambda \notin \operatorname{int}(\sigma_{\text{ess}}(K_0))$. Consider the case where a left-sided neighbourhood in $\mathbb{R}$ of λ is contained in $\rho(H_{0,+})$ and a right-sided

neighbourhood in $\mathbb{R}$ of λ is contained in $\rho(-H_{0,-})$. Since $\lambda \in \sigma_{\mathrm{ess}}(\pm H_{0,\pm})$ it follows from the assumption $\lambda \notin \mathrm{int}(\sigma_{\mathrm{ess}}(K_0)) = \mathrm{int}(\sigma_{\mathrm{ess}}(H_{0,+}) \cup \sigma_{\mathrm{ess}}(-H_{0,-}))$ that λ is an isolated point in the set $\sigma_{\mathrm{ess}}(H_{0,+})$ or in the set $\sigma_{\mathrm{ess}}(-H_{0,-})$. In the present situation it follows that $(\lambda, \lambda + \varepsilon) \cap \sigma(H_{0,+})$ or $(\lambda - \varepsilon, \lambda) \cap \sigma(-H_{0,-})$ consists of discrete eigenvalues that accumulate to λ. Now the statement follows again from Theorem A.5 (ii). $\qquad\square$

Remark 3.5. Note that (3.1), Theorem 3.1, and Theorem 3.4 remain true if instead of Hypothesis 2.3 one assumes that the operators $H_{0,+}$ and $H_{0,-}$ are both semibounded from below; equivalently one may assume that L_0 is semibounded from below.

We shall illustrate the statements in the Theorems 3.1 and 3.4 in the next example for a special case with q_0 chosen as a so-called finite-zone potential near $\pm\infty$; cf. [57, 58].

Example 3.6. Let $n_{\pm} \in \mathbb{N}$ and consider finitely many real numbers $\lambda_k^{\pm}$, $\lambda_{n_++1}^+$, $\mu_k^{\pm}$, $\mu_{n_-+1}^-$, $k \in \{1, \ldots, n_{\pm}\}$, such that

$$\lambda_1^+ < \mu_1^+ < \lambda_2^+ < \mu_2^+ < \cdots < \lambda_{n_+}^+ < \mu_{n_+}^+ < \lambda_{n_++1}^+,$$
$$\mu_{n_-+1}^- < \lambda_{n_-}^- < \mu_{n_-}^- < \cdots < \lambda_2^- < \mu_2^- < \lambda_1^- < \mu_1^-.$$

Let $(a,b) = \mathbb{R}$ and for r_0 in Hypothesis 2.1 we shall assume, in addition, that $r_0 = 1$ on (β, ∞) and $r_0 = -1$ on $(-\infty, \alpha)$. Moreover, we set $p_0 = 1$ on $(-\infty, \alpha)$ and (β, ∞). In other words, the differential expression ℓ_0 in (1.1) leads to operators $\pm H_{0,\pm}$ of the form

$$H_{0,+} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q_{0,+} \quad \text{and} \quad -H_{0,-} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - q_{0,-},$$

with Dirichlet boundary conditions at β (for $H_{0,+}$) and at α (for $-H_{0,-}$), and where $q_{0,\pm}$ denote the restrictions of q_0 onto (β, ∞) and $(-\infty, \alpha)$, respectively. According to [57, Chapter 8] there exist finite-zone potentials $q_{0,\pm}$ such that Hypothesis 2.2 is satisfied,

$$\sigma_{\mathrm{ess}}(H_{0,+}) = [\lambda_1^+, \mu_1^+] \cup [\lambda_2^+, \mu_2^+] \cup \cdots \cup [\lambda_{n_+}^+, \mu_{n_+}^+] \cup [\lambda_{n_++1}^+, \infty),$$
$$\sigma_{\mathrm{ess}}(-H_{0,-}) = (-\infty, \mu_{n_-+1}^-] \cup [\lambda_{n_-}^-, \mu_{n_-}^-] \cup \cdots \cup [\lambda_2^-, \mu_2^-] \cup [\lambda_1^-, \mu_1^-], \tag{3.6}$$

and $\pm H_{0,\pm}$ have at most finitely many discrete (real) eigenvalues.

According to Theorem 3.1 the nonreal spectrum of K_0 consists of discrete eigenvalues (with geometric multiplicity one) which are contained in a compact subset of $\mathbb{C}$. Possible accumulation points of nonreal eigenvalues are contained in the intersection of the two sets of bands of essential spectra in (3.6). Note that in the present situation possible points $\mu_i^+ = \lambda_j^-$ for some i, j or $\mu_l^- = \lambda_k^+$ for some l, k satisfy property (P) in Theorem 3.1 (ii) as $\pm H_{0,\pm}$ have at most finitely many discrete eigenvalues. Hence, if $\mu_{n_-+1}^- \leq \lambda_1^+$ and $\mu_1^- \leq \lambda_{n_++1}^+$ and

$$(\lambda_k^+, \mu_k^+) \cap (\lambda_l^-, \mu_l^-) = \emptyset \tag{3.7}$$

for all k, l, then K_0 has at most finitely many nonreal eigenvalues.

Concerning the real discrete eigenvalues Theorem 3.4 implies the following: real discrete eigenvalues of K_0 may only accumulate (from the left) to points $\lambda_k^+ = \lambda_l^-$

or (from the right) to points $\mu_i^+ = \mu_j^-$ for some k, l or i, j. Therefore, under the assumptions

$$\lambda_k^+ \neq \lambda_l^- \quad \text{and} \quad \mu_i^+ \neq \mu_j^- \tag{3.8}$$

for all k, l and i, j the operator K_0 has at most finitely many discrete real eigenvalues. Note that condition (3.7) implies (3.8) and hence it follows under the assumptions $\mu_{n_-+1}^- \leq \lambda_1^+$, $\mu_1^- \leq \lambda_{n_++1}^+$, and (3.7) that K_0 has at most finitely many discrete (real and nonreal) eigenvalues.

In the next theorem we discuss a more special situation that arises when, roughly speaking, the essential spectra of $H_{0,+}$ and $-H_{0,-}$ are separated by a gap. More precisely, if $\eta_\pm$ denote the lower bounds of the essential spectra of $H_{0,\pm}$,

$$\eta_+ := \min \sigma_{\text{ess}}(H_{0,+}) \quad \text{and} \quad \eta_- := \min \sigma_{\text{ess}}(H_{0,-}),$$

where $\eta_\pm := \infty$ if $\sigma_{\text{ess}}(H_{0,\pm}) = \emptyset$, then we shall assume that

$$-\eta_- < \eta_+.$$

In the following we fix some

$$\eta \in (-\eta_-, \eta_+). \tag{3.9}$$

Let H_0 be the operator in (1.2) and define the hermitian form $\langle \cdot, \cdot \rangle$ on $\text{dom} \, H_0$ for $f, g \in \text{dom} \, H_0$ by

$$\langle f, g \rangle := [(H_0 - \eta)f, g] = ((-H_{0,-} - \eta)f_-, f_-)_{L^2((a,\alpha);r_0)}$$

$$+ ((K_{0,\alpha\beta} - \eta)f_{\alpha\beta}, f_{\alpha\beta})_{L^2((\alpha,\beta);r_0)} + ((H_{0,+} - \eta)f_+, f_+)_{L^2((\beta,b);r_0)},$$

where f, g have the obvious decomposition with respect to (1.3) with $j = 0$, $f = f_- + f_{\alpha\beta} + f_+$ and $g = g_- + g_{\alpha\beta} + g_+$. Then the operator $H_0 - \eta$ has finitely many negative squares and the number κ of negative squares is given by the sum of the negative squares of the three entries on the diagonal, that is,

$$\kappa = \kappa_+ + \kappa_\eta + \kappa_-, \tag{3.10}$$

where

κ_η is the number of negative squares of the operator $K_{0,\alpha\beta} - \eta$,

κ_+ is the number of eigenvalues of the operator $-H_{0,-} - \eta$ in $(0, \infty)$,

κ_- is the number of eigenvalues of the operator $H_{0,+} - \eta$ in $(-\infty, 0)$.

This is due to the fact that the weight r_0 is negative on (a, α) and positive on (β, b). In the next theorem we discuss further properties of the eigenvalues of K_0 and their algebraic eigenspaces.

Theorem 3.7. *Assume Hypotheses 2.1, 2.2, and 2.3, and let $\eta_\pm, \eta$ be as above. Then the operator $K_0 - \eta$ has at most*

$$\kappa_0 := \kappa_+ + \kappa_\eta + \kappa_- + 2$$

negative squares which implies the following statements for the eigenvalues of K_0.

    (i) *There are at most κ_0 different real eigenvalues of K_0 with corresponding Jordan chains of length greater than one. The length of each of these chains is at most $2\kappa_0 + 1$.*

    (ii) *The nonreal spectrum of K_0 consists of at most κ_0 pairs $\{\mu_i, \overline{\mu}_i\}$, $\text{Im} \, \mu_i > 0$, of discrete eigenvalues with corresponding Jordan chains of length at most κ_0.*

Proof. The operator $H_0 - \eta$ has κ squares with κ as in (3.10). By (2.9) the resolvent difference of the resolvents of the operators H_0 and K_0 is a rank two operator and, hence, by Theorem A.3 (v) the operator $K_0 - \eta$ has at most $\kappa + 2$ negatives squares. Now the assertions in Theorem 3.7 follow from Theorem A.3.   □

Next we provide quantitative estimates on the number of eigenvalues of K_0 in the setting of Theorem 3.7. Recall that a finite rank perturbation of self-adjoint operators in Krein spaces may change the discrete spectrum dramatically, in particular, for general self-adjoint operators in (infinite dimensional) Krein spaces the number of eigenvalues in a gap of the essential spectrum may change arbitrarily under a rank one or rank two perturbation; cf. [13, Theorem 3.1]. The situation is different if the operators have finitely many negative squares; in the next example we illustrate how the estimates for rank one perturbations in [11] can be applied successively in the present situation to obtain upper bounds on the number of eigenvalues.

Example 3.8. Consider the case $\eta = 0$ in (3.9) and in Theorem 3.7, and fix an interval $I \subset \mathbb{R} \setminus \sigma_{\mathrm{ess}}(H_0)$ such that $I \subset (0, \eta_+)$ or $I \subset (-\eta_-, 0)$. In the following we denote the number of eigenvalues of a closed operator A in I by $n_A(I)$.

In the case $\alpha < \beta$ we make use of the auxiliary self-adjoint operator $K_{0,\alpha}$ in the Krein space $L^2((a, b); r_0)$, which is defined as the direct sum of the maximal realizations of ℓ_0 in $L^2((a, \alpha); r_0)$ and $L^2((\alpha, b); r_0)$ with Dirichlet boundary conditions at α. Then $\rho(K_{0,\alpha}) \neq \emptyset$ follows in the same way as in the proof of Theorem 2.5 from [12, Theorem 4.5] and since the resolvents of $K_{0,\alpha}$ and H_0 differ by a rank one operator we can apply [11, Corollary 3.2] to I and obtain

$$n_{K_{0,\alpha}}(I) \leq n_{H_0}(I) + n_{H_0,K_{0,\alpha}}(I) + 2\kappa + 3, \tag{3.11}$$

where $n_{H_0,K_{0,\alpha}}(I)$ stands for the number of joint eigenvalues of the operators H_0 and $K_{0,\alpha}$ in I and κ is the number of negative squares of H_0 in (3.10). Of course, $n_{H_0,K_{0,\alpha}}(I) \leq n_{H_0}(I)$ which gives

$$n_{K_{0,\alpha}}(I) \leq 2n_{H_0}(I) + 2\kappa + 3. \tag{3.12}$$

The same argument for the operators K_0 and $K_{0,\alpha}$, where one also uses that the operator $K_{0,\alpha}$ has at most $\kappa + 1$ negative squares by Theorem A.3 (v), leads to the estimate

$$n_{K_0}(I) \leq 2n_{K_{0,\alpha}}(I) + 2(\kappa + 1) + 3,$$

and with (3.12) we conclude

$$n_{K_0}(I) \leq 4n_{H_0}(I) + 4\kappa + 6 + 2(\kappa + 1) + 3 = 4n_{H_0}(I) + 6\kappa + 11. \tag{3.13}$$

Under the additional assumption $I \subset \rho(H_0)$ the estimate (3.13) improves and reduces to $n_{K_0}(I) \leq 6\kappa + 11$.

In the case $\alpha = \beta$ the above reasoning simplifies (see also Remark 2.4) and one can directly apply [11, Corollary 3.2] to the operators K_0 and H_0, so that instead of (3.11) we obtain immediately

$$n_{K_0}(I) \leq n_{H_0}(I) + n_{H_0,K_0}(I) + 2\kappa + 3, \tag{3.14}$$

where κ in (3.10) now has the form $\kappa = \kappa_+ + \kappa_-$. Note that under the additional assumption $I \subset \rho(H_0)$ the estimate (3.14) simplifies to $n_{K_0}(I) \leq 2\kappa + 3$.

Next we take a closer look at the discrete eigenvalues of K_0 in the situation of Corollary 2.6 and consider again the case $(a, b) = \mathbb{R}$, where the coefficients r_0, p_0, q_0 admit the limits

$$r_{\pm\infty} = \lim_{x \to \pm\infty} r_0(x), \; p_{\pm\infty} = \lim_{x \to \pm\infty} p_0(x), \quad \text{and} \quad q_{\pm\infty} = \lim_{x \to \pm\infty} q_0(x), \quad (3.15)$$

with $\pm r_{\pm\infty} > 0$, $p_{\pm\infty} > 0$, $q_{\pm\infty} \in \mathbb{R}$, and, in addition, we assume

$$q_{-\infty}/r_{-\infty} < q_\infty/r_\infty. \tag{3.16}$$

Then, by Corollary 2.6, the essential spectrum of K_0 is given by

$$\sigma_{\mathrm{ess}}(K_0) = \left(-\infty, \frac{q_{-\infty}}{r_{-\infty}}\right] \cup \left[\frac{q_\infty}{r_\infty}, \infty\right) \tag{3.17}$$

and by (3.16) there is a gap in (3.17), that is, the set $\sigma(K_0) \cap (q_{-\infty}/r_{-\infty}, q_\infty/r_\infty)$ consists of discrete eigenvalues. It is also clear from Theorem 3.7 that $K_0 - \eta$ with $q_{-\infty}/r_{-\infty} < \eta < q_\infty/r_\infty$ has finitely many negative squares, and hence the nonreal spectrum of K_0 consists of at most finitely many discrete eigenvalues.

We proceed by providing a Kneser type result for the indefinite Sturm–Liouville operator K_0, i.e., we obtain criteria for the accumulation/non-accumulation of eigenvalues of K_0 in $(q_{-\infty}/r_{-\infty}, q_\infty/r_\infty)$ to the edges of the essential spectrum (see, e.g., [65, Theorem 9.42 and Corollary 9.43] or [15, 51] for such type of results in the definite case). We first recall some notation from [15, 51]: the iterated (natural) logarithm $\log_n$ is defined recursively via

$$\log_0(x) := x, \qquad \log_n(x) := \log(\log_{n-1}(x)), \quad n \in \mathbb{N},$$

with the convention $\log(x) := \log|x|$ for negative values of x. Then $\log_n(x)$ will be continuous for $x > \mathrm{e}_{n-1}$ and positive for $x > \mathrm{e}_n$, where $\mathrm{e}_{-1} := -\infty$ and $\mathrm{e}_n := \mathrm{e}^{\mathrm{e}_{n-1}}$. Furthermore, abbreviate

$$L_n(x) := \frac{1}{\log'_{n+1}(x)} = \prod_{j=0}^{n} \log_j(x)$$

and

$$P_n(x) := \sum_{j=0}^{n-1} \frac{1}{L_j(x)} \quad \text{and} \quad Q_n(x) := -\frac{1}{4}\sum_{j=0}^{n-1} \frac{1}{L_j(x)^2}.$$

Here the convention $\sum_{j=0}^{-1} \equiv 0$ is used, so that, $P_0(x) = Q_0(x) = 0$.

Theorem 3.9. *Let $(a, b) = \mathbb{R}$ and assume that the coefficients r_0, p_0, q_0 admit the limits in (3.15) with $\pm r_{\pm\infty} > 0$, $p_{\pm\infty} > 0$, $q_{\pm\infty} \in \mathbb{R}$ such that (3.16) holds. For $n \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{R} \setminus [-\mathrm{e}_n, \mathrm{e}_n]$ let*

$$\Delta_{0,\pm}(x) := L_n(x)^2 \left(\frac{q_0(x)}{p_{\pm\infty}} - Q_n(x) \mp \frac{q_{\pm\infty}}{p_{\pm\infty} r_{\pm\infty}} r_0(x) + \frac{P_n(x)^2}{4}\left(1 - \frac{p_{\pm\infty}}{p_0(x)}\right)\right).$$

Then the set $\sigma(K_0) \cap (q_{-\infty}/r_{-\infty}, q_\infty/r_\infty)$ consists of discrete eigenvalues of K_0 which accumulate at $q_{\pm\infty}/r_{\pm\infty}$ if

$$\limsup_{x \to \pm\infty} \Delta_{0,\pm}(x) < -\frac{1}{4}$$

and do not accumulate at $q_{\pm\infty}/r_{\pm\infty}$ if

$$\liminf_{x \to \pm\infty} \Delta_{0,\pm}(x) > -\frac{1}{4}.$$

Proof. It follows from [15, Theorem 3.5] that $\sigma_{\mathrm{ess}}(H_{0,+}) = [q_\infty/r_\infty, \infty)$ and the set $\sigma(H_{0,+}) \cap (-\infty, q_\infty/r_\infty)$ consists of discrete eigenvalues of $H_{0,+}$ which accumulate at q_∞/r_∞ if

$$\limsup_{x \to \infty} \Delta_{0,+}(x) < -\frac{1}{4}$$

and do not accumulate at q_∞/r_∞ if

$$\liminf_{x \to \infty} \Delta_{0,+}(x) > -\frac{1}{4}.$$

In the same way one obtains that $\sigma_{\mathrm{ess}}(-H_{0,-}) = (\infty, q_{-\infty}/r_{-\infty})$ and the set $\sigma(-H_{0,-}) \cap (q_{-\infty}/r_{-\infty}, \infty)$ consists of discrete eigenvalues of $-H_{0,-}$ which accumulate at $q_{-\infty}/r_{-\infty}$ if

$$\limsup_{x \to -\infty} \Delta_{0,-}(x) < -\frac{1}{4}$$

and do not accumulate at $q_{-\infty}/r_{-\infty}$ if

$$\liminf_{x \to -\infty} \Delta_{0,-}(x) > -\frac{1}{4}.$$

The assumption (3.16) ensures that $q_\infty/r_\infty \notin \sigma_{\mathrm{ess}}(-H_{0,-})$ and $q_{-\infty}/r_{-\infty} \notin \sigma_{\mathrm{ess}}(H_{0,+})$ and hence the assertions follow directly from Theorem 3.4. $\square$

Remark 3.10. As mentioned above the operator $K_0 - \eta$ with $q_{-\infty}/r_{-\infty} < \eta < q_\infty/r_\infty$ has finitely many negative squares. In the case that the discrete eigenvalues of K_0 in $(q_{-\infty}/r_{-\infty}, q_\infty/r_\infty)$ do not accumulate to $q_{\pm\infty}/r_{\pm\infty}$ one may also choose $\eta = q_{\pm\infty}/r_{\pm\infty}$, that is, also the operator $K_0 - q_{\pm\infty}/r_{\pm\infty}$ has finitely many negative squares; cf. Theorem 3.7.

For the case $n = 0$ Theorem 3.9 reduces to the following statement.

Corollary 3.11. *Let $(a, b) = \mathbb{R}$ and assume that the coefficients r_0, p_0, q_0 admit the limits in (3.15) with $\pm r_{\pm\infty} > 0$, $p_{\pm\infty} > 0$, $q_{\pm\infty} \in \mathbb{R}$ such that (3.16) holds. Then the set $\sigma(K_0) \cap (q_{-\infty}/r_{-\infty}, q_\infty/r_\infty)$ consists of discrete eigenvalues of K_0 which accumulate at $q_{\pm\infty}/r_{\pm\infty}$ if*

$$\limsup_{x \to \pm\infty} x^2 \left(q_0(x) \mp \frac{q_{\pm\infty}}{r_{\pm\infty}} r_0(x) \right) < -\frac{p_{\pm\infty}}{4}$$

and do not accumulate at $q_{\pm\infty}/r_{\pm\infty}$ if

$$\liminf_{x \to \pm\infty} x^2 \left(q_0(x) \mp \frac{q_{\pm\infty}}{r_{\pm\infty}} r_0(x) \right) > -\frac{p_{\pm\infty}}{4}.$$

We end this section with the study of the point spectrum of the perturbed indefinite Sturm-Liouville operator K_1. Under the assumptions of Theorem 2.5 the indefinite Sturm-Liouville operator K_1 is self-adjoint in the Krein space $L^2((a, b); r_1)$ with $\rho(K_1) \neq \emptyset$ and the essential spectrum of K_1 is given by

$$\sigma_{\mathrm{ess}}(K_0) = \sigma_{\mathrm{ess}}(K_1) = \sigma_{\mathrm{ess}}(H_{1,+}) \cup \sigma_{\mathrm{ess}}(-H_{1,-}).$$

It is clear that the above considerations and results in this section also hold for K_1. In particular, the Kneser type results Theorem 3.9 and Corollary 3.11 can be formulated in the same way with the coefficients r_1, p_1, q_1. However, it is also possible to use information of the coefficients r_0, p_0, q_0 of the unperturbed operator K_0 to investigate possible accumulation of discrete eigenvalues of the perturbed operator K_1.

Theorem 3.12. *Let $(a,b) = \mathbb{R}$ and assume that the coefficients r_0, p_0, q_0 admit the limits (3.15) with $\pm r_{\pm\infty} > 0$, $p_{\pm\infty} > 0$, $q_{\pm\infty} \in \mathbb{R}$ such that (3.16) holds. Let $\Delta_{0,\pm}$ be as in Theorem 3.9 and assume that*

$$\lim_{x\to\pm\infty} L_n(x)^2 \left(|r_1(x) - r_0(x)| + P_n(x)^2 \left| \frac{1}{p_1(x)} - \frac{1}{p_0(x)} \right| + |q_1(x) - q_0(x)| \right) = 0$$
(3.18)

holds for some $n \in \mathbb{N}$. Then the following assertions hold.

(i) *The essential spectrum of K_1 is given by*

$$\sigma_{\mathrm{ess}}(K_1) = \left(-\infty, \frac{q_{-\infty}}{r_{-\infty}} \right] \cup \left[\frac{q_\infty}{r_\infty}, \infty \right),$$

the nonreal spectrum of K_1 consists of at most finitely many discrete eigenvalues, and $K_1 - \eta$ with $q_{-\infty}/r_{-\infty} < \eta < q_\infty/r_\infty$ has finitely many negative squares.

(ii) *The set $\sigma(K_1) \cap (q_{-\infty}/r_{-\infty}, q_\infty/r_\infty)$ consists of discrete eigenvalues of K_1 which accumulate at $q_{\pm\infty}/r_{\pm\infty}$ if*

$$\limsup_{x\to\pm\infty} \Delta_{0,\pm}(x) < -\frac{1}{4}$$

and do not accumulate at $q_{\pm\infty}/r_{\pm\infty}$ if

$$\liminf_{x\to\infty} \Delta_{0,\pm}(x) > -\frac{1}{4}.$$

Proof. Observe first that (3.18) implies

$$\lim_{x\to\infty} L_n(x)^2 |r_1(x) - r_0(x)| = 0,$$

$$\lim_{x\to\infty} L_n(x)^2 P_n(x)^2 \left| \frac{1}{p_1(x)} - \frac{1}{p_0(x)} \right| = 0,$$
(3.19)

$$\lim_{x\to\infty} L_n(x)^2 |q_1(x) - q_0(x)| = 0,$$

and since $\lim_{x\to\infty} L_n(x) = \infty$,

$$L_n(x)^2 P_n(x)^2 = \left(\sum_{j=0}^{n-1} \frac{L_n(x)}{L_j(x)} \right)^2,$$

and $L_n(x) \geq L_j(x)$ for $|x| > \mathrm{e}_n$, $j \in \{1, \dots, n-1\}$, it follows from (3.15) and (3.19) that

$$r_{\pm\infty} = \lim_{x\to\pm\infty} r_1(x), \quad p_{\pm\infty} = \lim_{x\to\pm\infty} p_1(x), \quad \text{and} \quad q_{\pm\infty} = \lim_{x\to\pm\infty} q_1(x).$$
(3.20)

Now the statement on the essential spectrum in (i) of the indefinite Sturm-Liouville operator K_1 follows from Corollary 2.6. It is a consequence of (3.20), (3.15), and [15, Theorem 3.2] that Hypotheses 2.1, 2.2, and 2.3 hold for the differential expression τ_1 with coefficients r_1, p_1, q_1. Hence Theorem 3.1 implies that the nonreal spectrum of K_1 consists of at most finitely many discrete eigenvalues and from Theorem 3.7 applied to K_1 we obtain that the operator $K_1 - \eta$ with $q_{-\infty}/r_{-\infty} < \eta < q_\infty/r_\infty$ has finitely many negative squares.

We now turn to the proof of the accumulation properties of the discrete eigenvalues of K_1 in (ii). For this, define (in the same way as $\Delta_{0,\pm}$) the functions

$$\Delta_{1,\pm}(x) := L_n(x)^2 \left(\frac{q_1(x)}{p_{\pm\infty}} - Q_n(x) \mp \frac{q_{\pm\infty}}{p_{\pm\infty}r_{\pm\infty}} r_1(x) + \frac{P_n(x)^2}{4} \left(1 - \frac{p_{\pm\infty}}{p_1(x)} \right) \right)$$

for $|x| > e_n$. It is easy to see that

$$\Delta_{1,\pm}(x) - \Delta_{0,\pm}(x) = L_n(x)^2 \left(\frac{q_1(x) - q_0(x)}{p_{\pm\infty}} \mp \frac{q_{\pm\infty}}{p_{\pm\infty}r_{\pm\infty}} (r_1(x) - r_0(x)) \right.$$

$$\left. + \frac{p_{\pm\infty} P_n(x)^2}{4} \left(-\frac{1}{p_1(x)} + \frac{1}{p_0(x)} \right) \right),$$

and from (3.19) we conclude

$$\lim_{x \to \pm\infty} \big(\Delta_{1,\pm}(x) - \Delta_{0,\pm}(x) \big) = 0.$$

Therefore, the limes superior and limes inferior of $\Delta_{1,\pm}$ and $\Delta_{0,\pm}$ at $\pm\infty$ coincide. Now assertion (ii) follows from Theorem 3.9 applied to K_1. □

Next we consider the case $n = 0$, which was excluded in the previous result, as one has to assume in this case, in addition, that p_1 and p_0 have the same limits at $\pm\infty$.

Corollary 3.13. *Let $(a,b) = \mathbb{R}$ and assume that the coefficients r_0, p_0, q_0 admit the limits (3.15) with $\pm r_{\pm\infty} > 0$, $p_{\pm\infty} > 0$, $q_{\pm\infty} \in \mathbb{R}$ such that (3.16) holds. If*

$$\lim_{x \to \pm\infty} x^2 \big(|r_1(x) - r_0(x)| + |q_1(x) - q_0(x)| \big) = 0 \quad and \quad \lim_{x \to \pm\infty} |p_1(x) - p_0(x)| = 0,$$

then (i) in Theorem 3.12 holds and the set $\sigma(K_1) \cap (q_{-\infty}/r_{-\infty}, q_{\infty}/r_{\infty})$ consists of discrete eigenvalues of K_1 which accumulate at $q_{\pm\infty}/r_{\pm\infty}$ if

$$\limsup_{x \to \pm\infty} x^2 \left(q_0(x) \mp \frac{q_{\pm\infty}}{r_{\pm\infty}} r_0(x) \right) < -\frac{p_{\pm\infty}}{4}$$

and do not accumulate at $q_{\pm\infty}/r_{\pm\infty}$ if

$$\liminf_{x \to \pm\infty} x^2 \left(q_0(x) \mp \frac{q_{\pm\infty}}{r_{\pm\infty}} r_0(x) \right) > -\frac{p_{\pm\infty}}{4}.$$

4. Periodic problems

In this section we study indefinite Sturm-Liouville operators with periodic coefficients near the singular endpoints $\pm\infty$ and we treat L^1-perturbations in the spirit of [16], see also [9, 28, 50, 61] for related considerations in the indefinite setting. More precisely, we shall assume that Hypothesis 2.1 holds and that the coefficients $1/p_0, q_0, r_0$ are ω-periodic in (β, ∞) and θ-periodic in $(-\infty, \alpha)$ for some $\omega, \theta > 0$; this also implies that ∞ and $-\infty$ are both in the limit point case, that is, Hypothesis 2.2 is automatically satisfied. In this situation the essential spectra of $H_{0,+}$ and $-H_{0,-}$ are purely absolutely continuous and consist of infinitely many closed intervals

$$\sigma_{\text{ess}}(H_{0,+}) = \bigcup_{k=1}^{\infty} [\lambda_k^+, \mu_k^+] \quad \text{and} \quad \sigma_{\text{ess}}(-H_{0,-}) = \bigcup_{l=1}^{\infty} [\lambda_l^-, \mu_l^-] \tag{4.1}$$

where the endpoints λ_k^+ and μ_k^+, $\lambda_k^+ < \mu_k^+$, denote the k-th eigenvalues of the regular Sturm–Liouville operator in $L^2((\beta, \beta+\omega); r_0)$ (in nondecreasing order) with periodic and semiperiodic boundary conditions, respectively, and $-\mu_l^-$ and $-\lambda_l^-$, $-\mu_l^- < -\lambda_l^-$, denote the l-th eigenvalues of the regular Sturm–Liouville operator in $L^2((\alpha-\theta, \alpha); r_0)$ (in nondecreasing order) with periodic and semiperiodic boundary conditions, respectively; cf. [22] or [68, Section 12].

Below we determine the essential spectra of indefinite Sturm operators K_0 and K_1, where K_0 is periodic near the endpoints and K_1 is an L^1-perturbation.

Theorem 4.1. *Assume that Hypothesis 2.1 holds and suppose that $1/p_0, q_0, r_0$ are ω-periodic in (β, ∞) and θ-periodic in $(-\infty, \alpha)$ for some $\omega, \theta > 0$. Then the following assertions hold.*

(i) *The indefinite Sturm-Liouville operator K_0 is self-adjoint in the Krein space $L^2(\mathbb{R}; r_0)$, the resolvent set $\rho(K_0)$ is nonempty, and the essential spectrum is given by*

$$\sigma_{\mathrm{ess}}(K_0) = \bigcup_{k=1}^{\infty} [\lambda_k^+, \mu_k^+] \cup \bigcup_{l=1}^{\infty} [\lambda_l^-, \mu_l^-]. \tag{4.2}$$

(ii) *If*

$$\int_{\mathbb{R}} \left(|r_1(t) - r_0(t)| + \left| \frac{1}{p_1(t)} - \frac{1}{p_0(t)} \right| + |q_1(t) - q_0(t)| \right) \, dt < \infty, \tag{4.3}$$

then also the indefinite Sturm-Liouville operator K_1 is self-adjoint in the Krein space $L^2(\mathbb{R}; r_1)$, the resolvent set $\rho(K_1)$ is nonempty, and the essential spectrum is given by

$$\sigma_{\mathrm{ess}}(K_1) = \bigcup_{k=1}^{\infty} [\lambda_k^+, \mu_k^+] \cup \bigcup_{l=1}^{\infty} [\lambda_l^-, \mu_l^-]. \tag{4.4}$$

Proof. (i) The periodicity of the coefficients $1/p_0, q_0, r_0$ implies the limit point condition at both singular endpoints $\pm\infty$. Hence the maximal operator L_0 in (2.3) is self-adjoint in the Hilbert space $L^2(\mathbb{R}; |r_0|)$. Define $H_{0,+}$ and $H_{0,-}$ as in (2.4). Then $H_{0,+}$ and $H_{0,-}$ are both semibounded from below; cf. [22] or [68]. This implies the semiboundedness of L_0. Hence we conclude from [12, Theorem 4.5] that the indefinite Sturm-Liouville operator K_0 is self-adjoint in the Krein space $L^2(\mathbb{R}; r_0)$ and that $\rho(K_0)$ is nonempty. Furthermore, taking into account (4.1) it follows that the essential spectrum of the block diagonal operator H_0 in (2.7) is given by

$$\sigma_{\mathrm{ess}}(H_0) = \sigma_{\mathrm{ess}}(H_{0,+}) \cup \sigma_{\mathrm{ess}}(-H_{0,-}) = \bigcup_{k=1}^{\infty} [\lambda_k^+, \mu_k^+] \cup \bigcup_{l=1}^{\infty} [\lambda_l^-, \mu_l^-].$$

Now the same perturbation argument as in the end of the proof of Theorem 2.5 leads to (4.2).

(ii) The assumption (4.3) together with [16, Theorem 2.1] implies that the operators $H_{1,-}$ and $H_{1,+}$ are self-adjoint and semibounded from below in $L^2((-\infty, \alpha); |r_1|)$ and $L^2((\beta, \infty); |r_1|)$, respectively. It then follows that also the maximal operator L_1 is self-adjoint and semibounded in the Hilbert space $L^2(\mathbb{R}; |r_1|)$ and we again conclude from [12, Theorem 4.5] that the indefinite Sturm-Liouville operator K_1 is self-adjoint in the Krein space $L^2(\mathbb{R}; r_1)$ and that $\rho(K_1)$ is nonempty. Since

$$\sigma_{\mathrm{ess}}(H_{0,+}) = \sigma_{\mathrm{ess}}(H_{1,+}) \quad \text{and} \quad \sigma_{\mathrm{ess}}(-H_{0,-}) = \sigma_{\mathrm{ess}}(-H_{1,-})$$

by (4.3) and [16, Theorem 2.1] we obtain $\sigma_{\mathrm{ess}}(H_0) = \sigma_{\mathrm{ess}}(H_1)$ and in the same way as in the end of the proof of Theorem 2.5 a perturbation argument leads to $\sigma_{\mathrm{ess}}(K_0) = \sigma_{\mathrm{ess}}(K_1)$, which finally shows (4.4). □

For the discrete spectrum of the operator K_0 we obtain the following corollary as an immediate consequence of Theorem 3.1, Theorem 3.4 (see also Remark 3.5), and Theorem 4.1 (i). Note that the periodic operators $\pm H_{0,\pm}$ have no discrete (real) eigenvalues and that in the present situation property (P) holds for points that satisfy $\lambda_k^+ = \mu_l^-$ or $\mu_k^+ = \lambda_l^-$ for some $k, l \in \mathbb{N}$; these points are automatically in $\mathrm{int}(\sigma_{\mathrm{ess}}(K_0))$.

Corollary 4.2. *Assume that Hypothesis 2.1 holds and suppose that $1/p_0, q_0, r_0$ are ω-periodic in (β, ∞) and θ-periodic in $(-\infty, \alpha)$ for some $\omega, \theta > 0$. Then the following assertions hold.*

(i) *The nonreal spectrum of K_0 consists of discrete eigenvalues with geometric multiplicity one which are contained in a compact subset of $\mathbb{C}$ and may only accumulate to points in*

$$[\lambda_k^+, \mu_k^+] \cap [\lambda_l^-, \mu_l^-], \qquad k, l \in \mathbb{N}.$$

Furthermore, the nonreal eigenvalues do not accumulate to points that satisfy $\lambda_k^+ = \mu_l^-$ or $\mu_k^+ = \lambda_l^-$ for some $k, l \in \mathbb{N}$.

(ii) *If $\lambda_k^+ \notin [\lambda_l^-, \mu_l^-]$ $(\mu_k^+ \notin [\lambda_l^-, \mu_l^-])$ for all $l \in \mathbb{N}$, then the real discrete eigenvalues of K_0 do not accumulate from the left at λ_k^+ (from the right at μ_k^+, respectively). In particular, if $\mu_1^- < \mu_i^+$, then $(\mu_k^+, \lambda_{k+1}^+) \cap \sigma_{\mathrm{ess}}(K_0) = \emptyset$ for all $k \geq i$ and there are at most finitely many discrete eigenvalues of K_0 in $(\mu_k^+, \lambda_{k+1}^+)$.*

(iii) *If $\lambda_l^- \notin [\lambda_k^+, \mu_k^+]$ $(\mu_l^- \notin [\lambda_k^+, \mu_k^+])$ for all $k \in \mathbb{N}$, then the real discrete eigenvalues of K_0 do not accumulate from the left at λ_l^- (from the right at μ_l^-, respectively). In particular, if $\lambda_j^- < \lambda_1^+$, then $(\mu_{l+1}^-, \lambda_l^-) \cap \sigma_{\mathrm{ess}}(K_0) = \emptyset$ for all $l \geq j$ and there are at most finitely many discrete eigenvalues of K_0 in $(\mu_{l+1}^-, \lambda_l^-)$.*

Note that the first statement in Corollary 4.2 (i) holds also for the perturbed operator K_1 in Theorem 4.1 (ii); this follows directly from Theorem 3.1, Remark 3.5, and Theorem 4.1 (ii). However, in order to exclude accumulation of eigenvalues of K_1 one has to impose a stronger "finite-first-moment" condition.

Theorem 4.3. *Assume that Hypothesis 2.1 holds and suppose that $1/p_0, q_0, r_0$ are ω-periodic in (β, ∞) and θ-periodic in $(-\infty, \alpha)$ for some $\omega, \theta > 0$. Assume, in addition, that*

$$\int_{\mathbb{R}} \left(|r_1(t) - r_0(t)| + \left| \frac{1}{p_1(t)} - \frac{1}{p_0(t)} \right| + |q_1(t) - q_0(t)| \right) |t| \, \mathrm{d}t < \infty. \qquad (4.5)$$

Then the following assertions hold.

(i) *The nonreal spectrum of K_1 consists of discrete eigenvalues with geometric multiplicity one which are contained in a compact subset of $\mathbb{C}$ and may only accumulate to points in*

$$[\lambda_k^+, \mu_k^+] \cap [\lambda_l^-, \mu_l^-], \qquad k, l \in \mathbb{N}.$$

Furthermore, the nonreal eigenvalues do not accumulate to points that satisfy $\lambda_k^+ = \mu_l^-$ or $\mu_k^+ = \lambda_l^-$ for some $k, l \in \mathbb{N}$.

(ii) If $\lambda_k^+ \notin [\lambda_l^-, \mu_l^-]$ ($\mu_k^+ \notin [\lambda_l^-, \mu_l^-]$) for all $l \in \mathbb{N}$, then the real discrete eigenvalues of K_1 do not accumulate from the left at λ_k^+ (from the right at μ_k^+, respectively). In particular, if $\mu_1^- < \mu_i^+$, then $(\mu_k^+, \lambda_{k+1}^+) \cap \sigma_{\mathrm{ess}}(K_1) = \emptyset$ for all $k \geq i$ and there are at most finitely many discrete eigenvalues of K_1 in $(\mu_k^+, \lambda_{k+1}^+)$.

(iii) If $\lambda_l^- \notin [\lambda_k^+, \mu_k^+]$ ($\mu_l^- \notin [\lambda_k^+, \mu_k^+]$) for all $k \in \mathbb{N}$, then the real discrete eigenvalues of K_1 do not accumulate from the left at λ_l^- (from the right at μ_l^-, respectively). In particular, if $\lambda_j^- < \lambda_1^+$, then $(\mu_{l+1}^-, \lambda_l^-) \cap \sigma_{\mathrm{ess}}(K_1) = \emptyset$ for all $l \geq j$ and there are at most finitely many discrete eigenvalues of K_1 in $(\mu_{l+1}^-, \lambda_l^-)$.

Proof. Observe first that condition (4.5) implies (4.3) and hence the indefinite Sturm-Liouville operator K_1 is self-adjoint in the Krein space $L^2(\mathbb{R}; r_1)$, the resolvent set $\rho(K_1)$ is nonempty, and the essential spectrum is given by (4.4). It follows from the condition (4.5) and [16, Theorem 2.3] that every gap in $\sigma_{\mathrm{ess}}(H_{1,+})$ and every gap in $\sigma_{\mathrm{ess}}(-H_{1,-})$ contains at most finitely many eigenvalues and hence the same is true for the operator H_1. Therefore property (P) in Theorem 3.1 (ii) holds for points that satisfy $\lambda_k^+ = \mu_l^-$ or $\mu_k^+ = \lambda_l^-$ for some $k, l \in \mathbb{N}$ and now (i) follows from Theorem 3.1. Furthermore, the same arguments as in the proof of Theorem 3.4 show that the discrete eigenvalues of K_1 do not accumulate to boundary points λ_k^+ or μ_k^+ of the essential spectrum, where $\lambda_k^+ \notin [\lambda_l^-, \mu_l^-]$ or $\mu_k^+ \notin [\lambda_l^-, \mu_l^-]$ for all $l \in \mathbb{N}$. This implies (ii); assertion (iii) follows in the same way (see also the discussion below Corollary 4.2). $\square$

Appendix A. Some facts on self-adjoint operators in Krein spaces

In this appendix we recall the concept of Krein spaces and some properties of certain classes of self-adjoint operators therein, which appear in this paper. We refer the interested reader to the monographs [1, 21, 32] for a thorough introduction to operator theory in Krein spaces.

In the following $(\mathcal{K}, (\cdot, \cdot))$ denotes a Hilbert space and J is a bounded self-adjoint operator in $\mathcal{K}$ with the property $J^2 = I$. Define a new inner product by

$$[x, y] := (Jx, y), \quad x, y \in \mathcal{K}.$$

Since $\sigma(J) \subset \{-1, 1\}$ we have the *fundamental decomposition* $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, where $\mathcal{K}_\pm = \ker(J \mp 1)$. Note that $\mathcal{K}_+$ and $\mathcal{K}_-$ are also orthogonal with respect to $[\cdot, \cdot]$ and that the inner product $[\cdot, \cdot]$ is *indefinite* if $\mathcal{K}_\pm \neq \{0\}$, that is, $[\cdot, \cdot]$ takes positive and negative values:

$$[x_+, x_+] > 0 \quad \text{and} \quad [x_-, x_-] < 0, \qquad x_\pm \in \mathcal{K}_\pm \setminus \{0\}.$$

The space $(\mathcal{K}, [\cdot, \cdot])$ is then called a *Krein space* and the operator J is called *fundamental symmetry*. We mention that often Krein spaces are introduced by starting with an indefinite inner product and a fundamental decomposition, see [1, 21, 32].

Let $(\mathcal{K}, [\cdot, \cdot])$ be a Krein space and let A be a bounded or unbounded (with respect to the Hilbert space norm) operator in $\mathcal{K}$. The adjoint A^+ is defined via the indefinite inner product $[\cdot, \cdot]$ and one has $A^+ = JA^*J$, where $*$ denotes the adjoint with respect to the Hilbert space scalar product $(\cdot, \cdot)$. It follows that A is self-adjoint with respect to $(\cdot, \cdot)$ if and only if JA is self-adjoint with respect to $[\cdot, \cdot]$. It is important to note that the spectral properties of operators which are self-adjoint with respect to a Krein space inner product differ essentially from the

ones of self-adjoint operators in Hilbert spaces, e.g., the spectrum is in general not real and may also coincide with $\mathbb{C}$. However, the indefiniteness of the inner product $[\cdot, \cdot]$ can be used to further classify eigenvalues of operators in Krein spaces, e.g. an isolated point $\lambda_0 \in \sigma_{\mathrm{p}}(A)$ is called of *positive* (*negative*) *type* if all corresponding eigenvectors x satisfy $[x, x] > 0$ ($[x, x] < 0$, respectively). This notion is extended to all points from the approximate point spectrum $\sigma_{\mathrm{ap}}(A)$ in the next definition. Recall that for a self-adjoint operator A in a Krein space all real spectral points belong to $\sigma_{\mathrm{ap}}(A)$; cf. [21, Corollary VI.6.2].

Definition A.1. *For a self-adjoint operator A in the Krein space $(\mathcal{K}, [\cdot, \cdot])$ a point $\lambda_0 \in \sigma(A)$ is called a spectral point of* positive (negative) *type of A if $\lambda_0 \in \sigma_{\mathrm{ap}}(A)$ and every sequence (x_n) in* $\mathrm{dom}\, A$ *with $\|x_n\| = 1$ and $\|(A - \lambda_0)x_n\| \to 0$ as $n \to \infty$ satisfies*

$$\liminf_{n \to \infty} [x_n, x_n] > 0 \quad \left(\limsup_{n \to \infty} [x_n, x_n] < 0, \; respectively \right).$$

Spectral theory in Krein spaces is often focusing on sign-type spectrum and further properties in a neighbourhood of such spectral points. The following proposition is of this nature; for a proof see [56] and also [2, 40].

Proposition A.2. *Let A be a self-adjoint operator in the Krein space $(\mathcal{K}, [\cdot, \cdot])$ and let λ_0 be a spectral point of positive type of A. Then λ_0 is real, the non-real spectrum of A cannot accumulate to λ_0, and there exists an open neighbourhood $\mathcal{U}$ in $\mathbb{C}$ of λ_0 such that the following statements hold.*

  (i) *All spectral points in $\mathcal{U} \cap \mathbb{R}$ are of positive type.*
  (ii) *There exists a number $M > 0$ such that*

$$\|(A - \lambda)^{-1}\| \leq \frac{M}{|\mathrm{Im}\, \lambda|} \quad \text{for all } \lambda \in \mathcal{U} \setminus \mathbb{R}.$$

  (iii) *There exists a local spectral function of A of positive type: To each interval δ with $\overline{\delta} \subset \mathcal{U}$ there exists a self-adjoint projection $E(\delta)$ which commutes with A, the space $(E(\delta)\mathcal{K}, [\cdot, \cdot])$ is a Hilbert space and $\sigma(A | E(\delta)\mathcal{K}) \subset \sigma(A) \cap \overline{\delta}$.*

An analogous statement holds for spectral points of negative type.

Roughly speaking, Proposition A.2 (iii) states that in a neighbourhood of a spectral point of positive type A behaves (locally) like a self-adjoint operator in a Hilbert space. In what follows we present two classes of operators with the property that they have intervals with spectrum of positive or negative type: operators with finitely many squares and locally definitizable operators.

We shall say that a self-adjoint operator A in a Krein space $(\mathcal{K}, [\cdot, \cdot])$ has κ *negative squares*, $\kappa \in \mathbb{N} \cup \{0\}$, if the hermitian form $\langle \cdot, \cdot \rangle$ on $\mathrm{dom}\, A$, defined by

$$\langle f, g \rangle := [Af, g], \qquad f, g \in \mathrm{dom}\, A,$$

has κ negative squares, that is, there exists a κ-dimensional subspace $\mathcal{M}$ in $\mathrm{dom}\, A$ such that $\langle v, v \rangle < 0$ if $v \in \mathcal{M}$, $v \neq 0$, but no $\kappa + 1$ dimensional subspace with this property. In the following theorem we recall some spectral properties of self-adjoint operators with finitely many negative squares. The statements are well known and are consequences of the general results in [25, 54, 55], see also [18, Theorem 3.1]. In item (iv) below we use the notation $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Theorem A.3. *Let A be a self-adjoint operator in the Krein space $(\mathcal{K}, [\cdot, \cdot])$ such that $\rho(A) \neq \emptyset$ and assume that A has κ negative squares. Then the following holds.*

(i) *The nonreal spectrum of A consists of at most κ pairs $\{\mu_i, \overline{\mu}_i\}$, $\operatorname{Im} \mu_i > 0$, of discrete eigenvalues. Denote for an eigenvalue λ of A the signature of the inner product $[\cdot, \cdot]$ on the algebraic eigenspace by $\{\kappa_-(\lambda), \kappa_0(\lambda), \kappa_+(\lambda)\}$. Then*

$$\sum_{\lambda \in \sigma_{\mathrm{p}}(A) \cap (-\infty, 0)} (\kappa_+(\lambda) + \kappa_0(\lambda)) + \sum_{\lambda \in \sigma_{\mathrm{p}}(A) \cap (0, \infty)} (\kappa_-(\lambda) + \kappa_0(\lambda)) + \sum_i \kappa_0(\mu_i) \leq \kappa, \quad \text{(A.1)}$$

and, if $0 \notin \sigma_{\mathrm{p}}(A)$, then equality holds in (A.1).

(ii) *There are at most κ different real nonzero eigenvalues of A with corresponding Jordan chains of length greater than one. The length of each of these chains is at most $2\kappa + 1$.*

(iii) *There exists a set Ξ consisting of at most κ real eigenvalues of A such that all spectral points in $(0, \infty) \setminus \Xi$ are of positive type and all spectral points in $(-\infty, 0) \setminus \Xi$ are of negative type with respect to A.*

(iv) *There exist an open neighbourhood $\mathcal{U}$ of $\overline{\mathbb{R}}$ in $\overline{\mathbb{C}}$ and $M > 0$ such that*

$$\|(A - \lambda)^{-1}\| \leq M \frac{(|\lambda| + 1)^{4\kappa + 2}}{|\operatorname{Im} \lambda|^{2\kappa + 2}} \quad \text{for all } \lambda \in \mathcal{U} \setminus \overline{\mathbb{R}}.$$

(v) *Let B be a self-adjoint operator in $(\mathcal{K}, [\cdot, \cdot])$ with $\rho(A) \cap \rho(B) \neq \emptyset$ and assume*

$$\dim\big(\operatorname{ran}\big((A - \lambda)^{-1} - (B - \lambda)^{-1}\big)\big) = n_0 < \infty$$

for some (and hence for all) $\lambda \in \rho(A) \cap \rho(B)$. Then B has $\widetilde{\kappa} \geq 0$ negative squares, where $|\widetilde{\kappa} - \kappa| \leq n_0$.

The second class of operators is the class of locally definitizable operators. They appeared first in a paper by H. Langer in 1967 [53] (without that name). Later, P. Jonas introduced the notion of locally definitizable operators, see, e.g, [38, 39, 40].

Definition A.4. *Let Ω be a domain in $\overline{\mathbb{C}}$ which is symmetric with respect to $\overline{\mathbb{R}}$ such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections with the open upper and lower half-plane are simply connected. Let A be a self-adjoint operator in the Krein space $(\mathcal{K}, [\cdot, \cdot])$ such that $\sigma(A) \cap (\Omega \setminus \overline{\mathbb{R}})$ consists of isolated points which are poles of the resolvent of A, and no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of the non-real spectrum of A. The operator A is called* definitizable over Ω *if the following holds.*

(i) *For every closed subset Δ of $\Omega \cap \overline{\mathbb{R}}$ there exist an open neighbourhood $\mathcal{U}$ of Δ in $\overline{\mathbb{C}}$ and numbers $m \geq 1$, $M > 0$ such that*

$$\|(A - \lambda)^{-1}\| \leq M \frac{(|\lambda| + 1)^{2m - 2}}{|\operatorname{Im} \lambda|^m} \quad \text{for all } \lambda \in \mathcal{U} \setminus \overline{\mathbb{R}}.$$

(ii) *For every $\lambda \in \Omega \cap \overline{\mathbb{R}}$ there exists an open connected neighbourhood I_λ in $\overline{\mathbb{R}}$ and two disjoint open intervals I', I'' with $I_\lambda \setminus \{\lambda\} = I' \cup I''$ and the following property: All spectral points in I' are either of positive or of negative type and all spectral points in I'' are either of positive or of negative type with respect to A.*

Obviously, Theorem A.3 implies that a self-adjoint operator in a Krein space with finitely many negative squares is definitizable over $\overline{\mathbb{C}}$. Operators definitizable

over $\overline{\mathbb{C}}$ are definitizable in the sense of H. Langer [55] (see also [40]), which is a well-studied class of operators in Krein spaces.

Roughly speaking, the property of an operator to be locally definitizable is stable under finite rank perturbations and the local sign type properties are preserved. A similar statement is valid for the accumulation of real discrete eigenvalues. This is the content of the next theorem from [4] and item (ii) from [10].

Theorem A.5. *Let Ω be as in Definition A.4 and let A and B be self-adjoint operators in a Krein space $(\mathcal{K}, [\cdot, \cdot])$ with $\rho(A) \cap \rho(B) \cap \Omega \neq \emptyset$, and assume that for some $\lambda_0 \in \rho(A) \cap \rho(B)$ the difference*

$$(A - \lambda_0)^{-1} - (B - \lambda_0)^{-1}$$

is a finite rank operator. Then the operator A is definitizable over Ω if and only if B is definitizable over Ω; in this case the following holds.

(i) *If $I \subset \Omega \cap \mathbb{R}$ is an open interval with boundary point $\lambda \in \Omega \cap \overline{\mathbb{R}}$ and the spectral points in I are of positive (negative) type with respect to A, then there exists an open interval I', $I' \subset I$, with boundary point λ such that the spectral points in I' are of positive (negative, respectively) type with respect to B.*

(ii) *Let J be an open interval such that $\overline{J} \subset \Omega \cap \overline{\mathbb{R}}$. Then $\sigma(A) \cap J$ consists of finitely many discrete eigenvalues if and only if $\sigma(B) \cap J$ consists of finitely many discrete eigenvalues.*

References

[1] T. Ya. Azizov and I. S. Iokhvidov, *Linear Operators in Spaces with an Indefinite Metric*, John Wiley and Sons, Chichester, New York, 1989.

[2] T. Ya. Azizov, P. Jonas, and C. Trunk, *Spectral points of type π_+ and π_- of selfadjoint operators in Krein spaces*, J. Funct. Anal. 226 (2005), 114–137.

[3] J. Behrndt, *On the spectral theory of singular indefinite Sturm-Liouville operators*, J. Math. Anal. Appl. 334 (2007), 1439–1449.

[4] J. Behrndt, *Finite rank perturbations of locally definitizable self-adjoint operators in Krein spaces*, J. Operator Theory 58 (2007), 415–440.

[5] J. Behrndt, *An open problem: Accumulation of nonreal eigenvalues of indefinite Sturm-Liouville operators*, Integral Equations Operator Theory, 77 (2013), 299–301.

[6] J. Behrndt, S. Chen, F. Philipp, and J. Qi, *Estimates on the non-real eigenvalues of regular indefinite Sturm-Liouville problems*, Proc. Roy. Soc. Edinburgh Sect. A 144 (2014), 1113–1126.

[7] J. Behrndt, Q. Katatbeh, and Carsten Trunk, *Accumulation of complex eigenvalues of indefinite Sturm-Liouville operators*, J. Phys. A: Math. Theor. 41 (2008), 244003.

[8] J. Behrndt, Q. Katatbeh, and C. Trunk, *Non-real eigenvalues of singular indefinite Sturm-Liouville operators*, Proc. Amer. Math. Soc. 137 (2009), 3797–3806.

[9] J. Behrndt, R. Möws, and C. Trunk, *Eigenvalue estimates for singular left-definite Sturm-Liouville operators*, J. Spectral Theory 1 (2011), 327–347.

[10] J. Behrndt, R. Möws, and C. Trunk, *On finite rank perturbation of selfadjoint operators in Krein spaces and eigenvalues in spectral gaps*, Complex Anal. Oper. Theory 8 (2014), 925–936.

[11] J. Behrndt, R. Möws, and C. Trunk, *Eigenvalue estimates for operators with finitely many negative squares*, Opuscula Math. 36 (2016), 717–734.

[12] J. Behrndt and F. Philipp, *Spectral analysis of singular ordinary differential operators with indefinite weights*, J. Differential Equations 248 (2010), 2015–2037.

[13] J. Behrndt and F. Philipp, *Finite rank perturbations in Pontryagin spaces and a Sturm-Liouville problem with λ-rational boundary conditions*, Operator Theory Advances Applications 263 (2018), 163–189.

[14] J. Behrndt, F. Philipp, and C. Trunk, *Bounds for the non-real spectrum of differential operators with indefinite weights*, Math. Ann. 357 (2013), 185–213.

[15] J. Behrndt, P. Schmitz, G. Teschl, and C. Trunk, *Relative oscillation theory and essential spectra of Sturm-Liouville operators*, J. Math. Anal. Appl. 518 (2023), Paper No. 126673

[16] J. Behrndt, P. Schmitz, G. Teschl, and C. Trunk, *Perturbations of periodic Sturm-Liouville operators*, Adv. Math. 422 (2023), 109022.

[17] J. Behrndt, P. Schmitz, and C. Trunk, *Spectral bounds for indefinite singular Sturm-Liouville operators with uniformly locally integrable potentials*, J. Differential Equations 267 (2019), 468–493.

[18] J. Behrndt and C. Trunk, *On the negative squares of indefinite Sturm-Liouville operators*, J. Differential Equations 238 (2007), 491–519.

[19] C. Bennewitz and W.N. Everitt, *On second-order left-definite boundary value problems*, Lect. Notes Math. 1032, Springer (1983), 31–67.

[20] P. Binding and M. Möller, *Negativity indices for definite and indefinite Sturm-Liouville problems*, Math. Nachr. 283 (2010), 180–192.

[21] J. Bognar, *Indefinite Inner Product Spaces*, Springer, Berlin, 1974.

[22] B. M. Brown, M. S. P. Eastham, and K. M. Schmidt, *Periodic Differential Operators*, Birkhäuser, Basel, 2013.

[23] S. Chen and J. Qi, *A priori bounds and existence of non-real eigenvalues of indefinite Sturm-Liouville problems*, J. Spectr. Theory 4 (2014), 53–63.

[24] S. Chen, J. Qi, and B. Xie, *The upper and lower bounds on the non-real eigenvalus of indefinite Sturm-Liouville problems*, Proc. Amer. Math. Soc. 144 (2016), 547–559.

[25] B. Ćurgus and H. Langer, *A Krein space approach to symmetric ordinary differential operators with an indefinite weight function*, J. Differential Equations 79 (1989), 31–61.

[26] B. Ćurgus and B. Najman, *The operator $sgn(x)\frac{d^2}{dx^2}$ is similar to a selfadjoint operator in $L^2(\mathbb{R})$*, Proc. Amer. Math. Soc. 123 (1995), 1125–1128.

[27] K. Daho and H. Langer, *Sturm-Liouville operators with an indefinite weight function*, Proc. Roy. Soc. Edinburgh Sect. A 78 (1977/78), 161–191.

[28] K. Daho and H. Langer, *Sturm-Liouville operators with an indefinite weight function: The periodic case*, Rad. Mat. 2 (1986), 165–188.

[29] N. Dunford and J. T. Schwartz, *Linear operators. Part II: Spectral Theory, Self Adjoint Operators in Hilbert Space*, John Wiley & Sons Ltd., 1988.

[30] F. Gesztesy and B. Simon *A short proof of Zheludev's theorem*, Trans. Amer. Math. Soc. 335 (1993), 329–340.

[31] F. Gesztesy and M. Ünal, *Perturbative oscillation criteria and Hardy-type inequalities*, Math. Nachr. 189 (1998), 121–144.

[32] A. Gheondea, *An Indefinite Excursion in Operator Theory: Geometric and Spectral Treks in Kreĭn Spaces*, Lond. Math. Soc. Lect. Note Ser. 476, Cambridge University Press, 2022.

[33] X. Guo, H. Sun, and B. Xie, *Non-real eigenvalues of symmetric Sturm-Liouville problems with indefinite weight functions*, Electron. J. Qual. Theory Differ. Equ. 2017 (2017), 1–14.

[34] H. van Haeringen, *The bound states for the symmetric shifted Coulomb potential*, J. Math. Phys. 19 (1978), 2165–2170.

[35] P. Hartman, *On the linear logarithmic-exponential differential equation of the second-order*, Am. J. Math. 70 (1948), 764–779.

[36] P. Hartman, *Some examples in the theory of singular boundary value problems*, Am. J. Math. 74 (1952), 107–126.

[37] E. Hille, *Nonoscillation theorems*, Trans. Amer. Math. Soc. 64 (1948), 234–252.

[38] P. Jonas, *On a class of selfadjoint operators in Krein space and their compact perturbations*, Integral Equations Operator Theory 11 (1988), 351–384.

[39] P. Jonas, *On a problem of the perturbation theory of selfadjoint operators in Krein spaces*, J. Operator Theory 25 (1991), 183–211.

[40] P. Jonas, *On locally definite operators in Krein spaces*, Spectral Theory and Applications, Theta Foundation (2003), 95–127.

[41] I.M. Karabash and A.S. Kostenko, *Indefinite Sturm-Liouville operators with the singular critical point zero*, Proc. Roy. Soc. Edinburgh Sect. A 138 (2008), 801–820.

[42] I.M. Karabash, A.S. Kostenko, and M.M. Malamud, *The similarity problem for J-nonnegative Sturm-Liouville operators*, J. Differential Equations 246 (2009), 964–997.

[43] I.M. Karabash and M.M. Malamud, *Indefinite Sturm-Liouville operators $(sgn\,x)(\frac{d^2}{dx^2} + q(x))$ with finite-zone potentials*, Oper. Matrices 1 (2007), 301–368.

[44] I.M. Karabash and C. Trunk, *Spectral properties of singular Sturm-Liouville operators*, Proc. Roy. Soc. Edinburgh Sect. A 139 (2009), 483–503.

[45] M. Kikonko and A.B. Mingarelli, *Bounds on real and imaginary parts of non-real eigenvalues of a non-definite Sturm-Liouville problem*, J. Differential Equations 261 (2016), 6221–6232.

[46] A. Kneser, *Untersuchungen über die reellen Nullstellen der Integrale linearer Differentialgleichungen*, Math. Ann. 42 (1893), 409–435.

[47] Q. Kong, M. Möller, H. Wu, and A. Zettl, Indefinite Sturm-Liouville problems, Proc. Roy. Soc. Edinburgh Sect. A 133 (2003), 639–652.

[48] Q. Kong, H. Wu, and A. Zettl, Left-definite Sturm-Liouville problems, J. Differential Equations, 177 (2001), 1–26.

[49] Q. Kong, H. Wu, and A. Zettl, Singular left-definite Sturm-Liouville problems, J. Differential Equations 206 (2004), 1–29.

[50] A. Kostenko, *The similarity problem for indefinite Sturm-Liouville operators with periodic coefficients*, Oper. Matrices 5 (2011), 707–722.

[51] H. Krüger and G. Teschl, *Effective Prüfer angles and relative oscillation criteria*, J. Differential Equations 245 (2008), 3823–3848.

[52] H. Krüger and G. Teschl, *Relative oscillation theory, zeros of the Wronskian, and the spectral shift function*, Comm. Math. Phys. 287 (2009), 613–640.

[53] H. Langer, *Spektralfunktionen einer Klasse J-selbstadjungierter Operatoren*, Math. Nachr. 33 (1967), 107–120.

[54] H. Langer, *Spektraltheorie linearer Operatoren in J-Räumen und einige Anwendungen auf die Schar $L(\lambda) = \lambda^2 + \lambda B + C$*, Habilitationsschrift, Technische Universität Dresden, 1965.

[55] H. Langer, *Spectral Functions of Definitizable Operators in Krein Spaces*, in: Functional Analysis Proceedings of a Conference held at Dubrovnik, Yugoslavia, November 2-14, 1981, Lecture Notes in Mathematics 948, Springer Verlag Berlin-Heidelberg-New York (1982), pp. 1-46.

[56] H. Langer, A.S. Markus and V.I. Matsaev, *Locally definite operators in indefinite inner product spaces*, Math. Ann. 308 (1997), 405–424.

[57] B.M. Levitan, *Inverse Sturm-Liouville Problems*, Nauka, Moscow 1984 [Russian]; Engl. transl.: VNU Science Press, Utrecht, 1987.

[58] B.M. Levitan and I.S. Sargsjan, *Sturm-Liouville and Dirac Operators*, Nauka, Moscow, 1988 [Russian]; Engl. transl.: Kluwer, Dordrecht 1990

[59] M. Levitin and M. Seri, *Accumulation of complex eigenvalues of an indefinite Sturm-Liouville operator with a shifted Coulomb potential*, Oper. Matrices 10 (2016), 223–245.

[60] A.B. Mingarelli, *A survey of the regular weighted Sturm-Liouville problem - The non-definite case*, pp. 109–137, International Workshop on Applied Differential Equations (Beijing, 1985), World Scientific, Singapore, 1986.

[61] F. Philipp, *Indefinite Sturm-Liouville operators with periodic coefficients*, Oper. Matrices. 7 (2013), 777–811.

[62] J. Qi and B. Xie, *Non-real eigenvalues of indefinite Sturm-Liouville problems*, J. Differential Equations 255 (2013), 2291–2301.

[63] F. S. Rofe-Beketov, *A finiteness test for the number of discrete levels which can be introduced into the gaps of the continuous spectrum by perturbations of a periodic potential*, Doklady Akademii Nauk 156 (1964), 515–518.

[64] K.M. Schmidt, *Critical coupling constants and eigenvalue asymptotics of perturbed periodic Sturm–Liouville operators*, Comm. Math. Phys. 211 (2000), 465–485.

[65] G. Teschl, *Mathematical Methods in Quantum Mechanics; With Applications to Schrödinger Operators*, 2nd. ed., Graduate Studies in Mathematics 157, Amer. Math. Soc., Providence, 2014.

[66] C. Trunk, *Locally definitizable operators: The local structure of the spectrum*, in: Operator Theory, D. Alpay (Ed.), 241–259, Springer 2015.

[67] H. Weber, *Die Partiellen Differential-Gleichungen der Mathematischen Physik*, vol. 2, Vieweg, Braunschweig, 1912.

[68] J. Weidmann, Spectral Theory of Ordinary Differential Operators, Lecture Notes in Math. 1258, Springer, 1987.

[69] A. Zettl, *Sturm-Liouville theory*, Mathematical Surveys and Monographs 121. Amer. Math. Soc., Providence, 2005.

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