Proseminar Advanced Partial Differential Equations

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WS2021/22

Please see the lecture notes for further details.

1. Discuss the Helmholz equation on $\mathbb{R}^n$.
2. Show that the definition of the Fourier transform on L^2 in (8.6) is well defined (i.e., the limit exists and is independent of the sequence). Show that the Plancherel identity continues to hold.
3. Show
$$\int_0^\infty \frac{\sin(x)^2}{x^2} dx = \frac{\pi}{2}.$$
(Hint: Problem 4.1 (i) from the lecture notes.)
4. Provide the details for Example 8.1.
5. Suppose $f \in L^2(\mathbb{R}^n)$ show that $\varepsilon^{-1}(f(x+e_j\varepsilon)-f(x)) \to g_j(x)$ in L^2 if and only if $k_j\hat{f}(k) \in L^2$, where e_j is the unit vector into the j'th coordinate direction. Moreover, show $g_j = \partial_j f$ if $f \in H^1(\mathbb{R}^n)$.
6. Assume $g \in L^2(\mathbb{R}^n)$. Show that $\hat{u}(t) = \hat{g}(k)\mathrm{e}^{-t|k|^2}$ is differentiable and solves $\frac{d}{dt}\hat{u}(t)(k) = -|k|^2\hat{u}(t)(k)$ for $t > 0$. (Hint: $|\mathrm{e}^{-\varepsilon|k|^2} - 1| \le \varepsilon|k|^2$ for $\varepsilon \ge 0$.)
7. Consider $f(x) = \sqrt{x}$, $U = (0,1)$. Compute the weak derivative. For which p is $f \in W^{1,p}(U)$?
8. The class of absolutely continuous functions can be defined as the class of antiderivatives of integrable functions
$$AC[a,b] := \{f(x) = f(a) + \int_a^x h(y)dy | h \in L^1(a,b)\},$$
where $a < b$ are some real numbers. It is easy to see that every absolutely continuous function is in particular continuous, $AC[a,b] \subset C[a,b]$. Moreover, using Lebesgue's differentiation theorem one can show that an absolutely continuous function is differentiable a.e. with $f'(x) = h(x)$ (and hence h is uniquely defined a.e.). However, not every continuous function is absolutely continuous.

   Show that for $f, g \in AC[a,b]$ we have the integration by parts formula
$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx.$$

and hence every absolutely continuous functions has a weak derivative which equals the a.e. derivative. Show that the converse also holds.

(Hint: To show the integration by parts formula insert the definition on the left and use Fubini. To show that a weakly differentiable function is absolutely continuous, use Lemma B.15 to conclude that a weakly differentiable function is the antiderivative of its weak derivative.)

9. Consider $U := B_1(0) \subset \mathbb{R}^n$ and $f(x) = \tilde{f}(|x|)$ with $\tilde{f} \in C^1(0,1]$. Then $f \in W^{1,p}_{loc}(B_1(0) \setminus \{0\})$ and

$$\partial_j f(x) = \tilde{f}'(|x|)\frac{x_j}{|x|}.$$

Show that if $\lim_{r\to 0} r^{n-1}\tilde{f}(r) = 0$ then $f \in W^{1,p}(B_1(0))$ if and only if $\tilde{f}, \tilde{f}' \in L^p((0,1), r^{n-1}dr)$.

Conclude that for $f(x) := |x|^{-\gamma}$, $\gamma > 0$, we have $f \in W^{1,p}(B_1(0))$ with

$$\partial_j f(x) = -\frac{\gamma x_j}{|x|^{\gamma+2}}$$

provided $\gamma < \frac{n-p}{p}$. (Hint: Use integration by parts on a domain which excludes $B_\varepsilon(0)$ and let $\varepsilon \to 0$.)

10. Suppose $f \in W^{k,p}(U)$ and $h \in C^k_b(U)$. Then $h \cdot f \in W^{k,p}(U)$ and we have Leibniz' rule

$$\partial_\alpha(h \cdot f) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (\partial_\beta h)(\partial_{\alpha-\beta} f),$$

where $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}$, $\alpha! := \prod_{j=1}^m (\alpha_j!)$, and $\beta \le \alpha$ means $\beta_j \le \alpha_j$ for $1 \le j \le m$.

11. Suppose $f \in W^{1,p}(U)$ satisfies $\partial_j f = 0$ for $1 \le j \le n$. Show that f is constant if U is connected.

12. Suppose $f \in W^{1,p}(U)$. Show that $|f| \in W^{1,p}(U)$ with

$$\partial_j|f|(x) = \frac{\mathrm{Re}(f(x))}{|f(x)|}\partial_j \mathrm{Re}(f(x)) + \frac{\mathrm{Im}(f(x))}{|f(x)|}\partial_j \mathrm{Im}(f(x)),$$

In particular $\big|\partial_j|f|(x)\big| \le |\partial_j f(x)|$. Moreover, if f is real-valued we also have $f_\pm := \max(0, \pm f) \in W^{1,p}(U)$ with

$$\partial_j f_\pm(x) = \begin{cases} \pm\partial_j f(x), & \pm f(x) > 0, \\ 0, & \text{else}, \end{cases} \qquad \partial_j|f|(x) = \begin{cases} \partial_j f(x), & f(x) > 0, \\ -\partial_j f(x), & f(x) < 0, \\ 0, & \text{else}. \end{cases}$$

(Hint: $|f| = \lim_{\varepsilon\to 0} g_\varepsilon(\mathrm{Re}(f), \mathrm{Im}(f))$ with $g_\varepsilon(x,y) = \sqrt{x^2+y^2+\varepsilon^2} - \varepsilon$.)

13. Show that $W^{k,p}(U) \cap W^{j,q}(U)$ (with $1 \le p,q \le \infty$, $j,k \in \mathbb{N}_0$) together with the norm $\|f\|_{W^{k,p}\cap W^{j,q}} := \|f\|_{W^{k,p}} + \|f\|_{W^{j,q}}$ is a Banach space.

14. Show $W^{k,p}_0(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$ for $1 \le p < \infty$. (Hint: Consider $f\zeta_m$ with $\zeta_m \in C^\infty_c(\mathbb{R}^m)$ and $\zeta_m = 1$ on $B_m(0)$.)

15. Show that $f \in W_0^{k,p}(U)$ can be extended to a function $\bar{f} \in W_0^{k,p}(\mathbb{R}^n)$ by setting it equal to zero outside U. In this case the weak derivatives of $\bar{f}$ are obtained by setting the weak derivatives of f equal to zero outside U.

16. Suppose $\gamma \geq 1$. Show that $f \in W^{1,p}(U)$ implies $|f|^\gamma \in W^{1,p/\gamma}(U)$ with $\partial_j |f|^\gamma = \gamma |f|^{\gamma-1} \partial_j |f|$. (Hint: Problem 12.)

17. Let $1 \leq p < \infty$ and U bounded. Show that $Tf = f\big|_{\partial U}$ defined on $C(\overline{U}) \subseteq L^p(U) \to L^p(\partial U)$ is unbounded (and hence has no meaningful extension to $L^p(U)$). (Hint: Take a sequence which equals 1 on the boundary and converges to 0 in the interior.)

18. Show that the inequality $\|f\|_q \leq C \|\nabla f\|_p$ for $f \in W^{1,p}(\mathbb{R}^n)$ can only hold for $q = \frac{np}{n-p}$. (Hint: Consider $f_\lambda(x) = f(\lambda x)$.)

19. Show that $f(x) := \log \log(1 + \frac{1}{|x|})$ is in $W^{1,n}(B_1(0))$ if $n > 1$. (Hint: Problem 9.)

20. Consider $U := \{(x, y) \in \mathbb{R}^2 | 0 < x, y < 1,\ x^\beta < y\}$ and $f(x, y) := y^{-\alpha}$ with $\alpha, \beta > 0$. Show $f \in W^{1,p}(U)$ for $p < \frac{1+\beta}{(1+\alpha)\beta}$. Now observe that for $0 < \beta < 1$ and $\alpha < \frac{1-\beta}{2\beta}$ we have $2 < \frac{1+\beta}{(1+\alpha)\beta}$.

21. Prove Young's inequality
$$\alpha^{1/p} \beta^{1/q} \leq \frac{1}{p}\alpha + \frac{1}{q}\beta, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad \alpha, \beta \geq 0.$$
Show that equality occurs precisely if $\alpha = \beta$. (Hint: Take logarithms on both sides.)

22. Show that if $f \in L^{p_0} \cap L^{p_1}$ for some $p_0 < p_1$ then $f \in L^p$ for every $p \in [p_0, p_1]$ and we have the Lyapunov inequality
$$\|f\|_p \leq \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta},$$
where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\theta \in (0, 1)$. (Hint: Generalized Hölder inequality — see Problem B.12 from the notes.)

23. Let $U = B_1(0) \subset \mathbb{R}^n$ and consider
$$u_m(x) = \begin{cases} m^{\frac{n}{p}-1}(1 - m|x|), & |x| < \frac{1}{m}, \\ 0, & \text{else}. \end{cases}$$
Show that u_m is bounded in $W^{1,p}(U)$ for $1 \leq p < n$ but has no convergent subsequence in $L^{p^*}(U)$. (Hint: The beta integral might be useful — see Problem A.8 from the notes.)

24. Compute J^* for $U := (0, 1) \subset \mathbb{R}$ (defined in the notes in (10.7)).

25. Investigate the Helmholtz equation
$$\begin{aligned} -\Delta u(x) + u(x) &= f(x), & x &\in U, \\ u(x) &= 0, & x &\in \partial U, \end{aligned}$$
on a domain $U \subseteq \mathbb{R}^n$. (Note, that U is not required to be bounded.)

26. Find a weak formulation of the Poisson problem with Robin boundary conditions

$$-\Delta u(x) + \lambda u(x) = f(x), \qquad x \in U,$$
$$\frac{\partial u}{\partial \nu}(x) + a(x)u(x) = 0, \qquad x \in \partial U,$$

on a bounded domain $U \subseteq \mathbb{R}^n$ with a C^1 boundary. Here $a \in L^\infty(U, \mathbb{R})$. Establish existence of weak solutions for $\lambda > \lambda_0$. Show that if $a \geq 0$ is nonzero and U is bounded and connected, then all eigenvalues of the Laplacian with Robin boundary conditions are positive. (Hint: Green's first identity.)

27. Consider the Dirichlet problem $-\Delta u = 0$ on the punctured disc $U := B_1(0) \setminus \{0\} \subset \mathbb{R}^n$ with boundary data $g(x) = 0$ for $|x| = 1$ and $g(0) = 1$. Since this domain does not have a trace operator, we understand the boundary condition as $u - \bar{g} \in H_0^1(U)$, where $\bar{g} = 1 - |x|^2$. Find the corresponding weak solution. (Hint: Observe that the weak solution must be radial. In particular, you are looking for a radial harmonic function satisfying the boundary conditions.)

28. Consider a function $F : \mathbb{R} \to \mathbb{R}$ such that $|F(t)| \leq |t|^3$ for all $t \in \mathbb{R}$ and let $U \subset \mathbb{R}^3$ be a bounded domain with the extension property. Prove that if $u \in H^1(U)$ is a weak solution of the nonlinear Poisson equation $-\Delta u = F(u)$, then in fact we have $u \in H^2_{loc}(U)$. (Hint: Corollary 9.23.)

29. Let $U \subset \mathbb{R}^n$ be an open set with a bounded C^k boundary. Show that a function $f \in C^k(\partial U)$ has an extension $\bar{f} \in C^k_b(\overline{U})$ such that $\bar{f}|_{\partial U} = f$. (Hint: Reduce it to the case of a flat boundary.)

30. Recall (B.27) and (9.10). Show

$$\int_U v(D_l^{-\varepsilon} u) d^n x = -\int_U (D_l^\varepsilon v) u \, d^n x$$

as well as

$$D_l^\varepsilon(uv) = (T_{\varepsilon\delta^l} v)(D_l^\varepsilon u) + (D_l^\varepsilon v)u.$$

31. Let a be a coercive and symmetric bilinear form. Show that the solution of (10.40) is also the unique minimizer of

$$v \mapsto \frac{1}{2} a(v, v) - \langle v, f \rangle.$$

(Hint: Inspect the proof in Section 5.5.)

32. Show the maximum principle for weak solutions: Suppose either $4\theta c_1 > b_0^2$ or $b_0 = c_1 = 0$. Let $u, v \in H^1(U, \mathbb{R})$ with u a weak solution and v a weak subsolution. Then $v \leq u$ on ∂U implies $v \leq u$ on U.

33. Show that for a weak solution $u \in H^1(U)$ we have

$$\|\nabla u\|_2 \leq \varepsilon \|f\|_2 + C\|u\|_2.$$

(Hint: Use ellipticity and start from $\theta \|\nabla u\|_2^2 \leq \dots$.)

34. Let X be a Banach algebra. Show that if $f, g \in C^1(I, X)$ then $fg \in C^1(I, X)$ and $\frac{d}{dt} fg = \dot{f}g + f\dot{g}$.

35. Let $A : \mathfrak{D}(A) \subseteq X \to X$ be a closed operator. Show that

$$A \int_a^b f(t)dt = \int_a^b A\, f(t)dt.$$

holds for $f \in C(I, X)$ with $\mathrm{Ran}(f) \subseteq \mathfrak{D}(A)$ and $Af \in C(I, X)$.

36. Let X be a Hilbert space and $A \in \mathscr{L}(X)$. Show that $T(t)^*$ is a uniformly continuous operator group whose generator is A^*. Conclude that if A is skew adjoint, that is, $A^* = -A$, then T is unitary.

37. Discuss the discrete Schrödinger equation

$$\mathrm{i}\dot{u} = Hu, \qquad (Hu)_n := u_{n+1} + u_{n-1} + q_n u_n,$$

in $\ell^2(\mathbb{Z})$, where $q \in \ell^\infty(\mathbb{Z}, \mathbb{R})$. In particular, show $\|u(t)\| = \|u(0)\|$ and $\langle u(t), Hu(t)\rangle = \langle u(0), Hu(0)\rangle$.

38. Let $T(t)$ be a C_0-semigroup. Show that if $T(t_0)$ has a bounded inverse for one $t_0 > 0$ then this holds for all $t > 0$ and it extends to a strongly continuous group via $T(t) := T(-t)^{-1}$ for $t < 0$.

39. Consider the translation group $T(t) := T_t$ on $L^p(\mathbb{R})$, $1 \le p < \infty$. Show that this is a strongly continuous group and compute its generator. Show that it is not strongly continuous for $p = \infty$. (Hint: Problem B.15 in the notes.)

40. Let A be the generator of a C_0-semigroup $T(t)$. Show

$$T(t)f = f + tAf + \int_0^t (t-s)T(s)A^2 f\, ds, \qquad f \in \mathfrak{D}(A^2).$$

41. Let A be the generator of a C_0-semigroup $T(t)$ satisfying $\|T(t)\| \le M$. Derive the abstract Landau inequality

$$\|Af\| \le 2M\|A^2 f\|^{1/2}\|f\|^{1/2}.$$

(Hint: Problem 40.)

42. Let $T(t)$ be a C_0-semigroup and $\alpha > 0$, $\lambda \in \mathbb{C}$. Show that $S(t) := \mathrm{e}^{\lambda t}T(\alpha t)$ is a C_0-semigroup with generator $B = \alpha A + \lambda$, $\mathfrak{D}(B) = \mathfrak{D}(A)$.

43. Show that A generates a C_0 group of isometries, that is, $\|T(t)g\| = \|g\|$ for all $g \in X$ if and only if both A and $-A$ generate contraction semigroups. That is, both A and $-A$ satisfy the hypothesis of either the Hill–Yosida or the Lumer–Phillips theorem.

44. Let $T(t)$ be a contraction C_0-semigroup with generator A and $B \in \mathscr{L}(X)$. Show that $A+B$ generates a C_0-semigroup $S(t)$ satisfying $\|S(t)\| \le \mathrm{e}^{\|B\|t}$. (Hint: Use Problem B.18.)

45. Let $X = \ell^2(\mathbb{N})$ and $(Aa)_n := \mathrm{i}n^2 a_n$, $(Ba)_n := n a_n$ both defined maximally. Show that A generates a C_0-semigroup but $A+\varepsilon B$ does not for any $\varepsilon > 0$.

46. Consider the heat equation (Example 11.14) on $[0,1]$ with Neumann boundary conditions $u'(0) = u'(1) = 0$.

47. Show that a solution of the heat equation satisfies

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(t)\|^2 ds = \|u(0)\|_2^2.$$

48. Let $U \subseteq \mathbb{R}^n$ be a domain (not necessarily bounded). Consider $H_0^1(U) \oplus L^2(U)$ with norm

$$\|\xi\|^2 := \|u\|_{H^1}^2 + \|v\|_{L^2}^2, \qquad \xi = (u,v).$$

Show that A defined in (11.69), (11.70) generates a C_0-group.

49. Discuss the telegraph equation

$$u_{tt} + b\,u_t = \Delta u + cu,$$

where $c, b \in L^\infty(U)$, on a bounded domain with Dirichlet boundary conditions. (Hint: Problem 44.)

50. Show that mild solutions of the semilinear problem

$$\dot u = Au + F(u), \qquad u(0) = g,$$

with F Lipschitz on bounded sets are global if $\|F(x)\| \le C(1 + \|x\|)$ for some constant C. (Hint: Use Gronwall's inequality to bound $\|u(t)\|$.)

51. Let $u \in C([-t_0,t_0], H^{r+2}(\mathbb{R}^n)) \cap C^1([-t_0,t_0], H^r(\mathbb{R}^n))$ be a strong solution of the NLS equation (with $r > \frac{n}{2}$). Show that momentum and energy are independent of $t \in [-t_0, t_0]$.

52. Show that the real derivative (with respect to the identification $\mathbb{C} \cong \mathbb{R}^2$) of $F(u) = |u|^{\alpha-1}u$ is given by

$$F'(u)v = |u|^{\alpha-1}v + (\alpha-1)|u|^{\alpha-3}u\mathrm{Re}(u^*v).$$

Conclude in particular,

$$|F'(u)v| \le \alpha|u|^{\alpha-1}|v|, \qquad |F(u) - F(v)| \le \alpha(|u|^{\alpha-1} + |v|^{\alpha-1})|u-v|.$$

Moreover, the second derivative is given by

$$vF''(u)w = (\alpha-1)|u|^{\alpha-5}u\big((\alpha+1)\mathrm{Re}(u^*v)\mathrm{Re}(u^*w) - u^2v^*w^*\big).$$

and hence

$$|vF''(u)w| \le (\alpha-1)(\alpha+2)|u|^{\alpha-2}|v||w|.$$

53. Let $f \in H^1(\mathbb{R})$. Show $\|f\|_\infty^2 \le 2\|f\|_2\|f'\|_2$ and hence $\|f\|_\infty \le \|f\|_{1,2}$.

54. Show that if $u \in H^2(\mathbb{R}) \cap L^2(\mathbb{R}, x^4 dx)$, then $xu'(x) \in L^2$.

55. Let L be self-adjoint with an orthonormal basis of eigenfunctions w_j corresponding to the eigenvalues E_j. For a complex-valued function F define

$$F(L)g := \sum_{j=0}^{\infty} F(E_j)\langle w_j, g\rangle w_j.$$

Show

$$\|F(L)\| = \sup_{j\in\mathbb{N}_0} |F(E_j)|.$$

56. Let L be as in the previous problem with $E_j \geq E_0 > 0$. Show that the semigroup $T(t)$ generated by $A := -L$

$$\|(T(t)-1)f\| \leq Ct\|Af\|, \quad \|LT(t)\| \leq \frac{C}{t}, \qquad f \in \mathfrak{D}(A),\ 0 < t \leq 1.$$

57. Show that a differentiable semigroup satisfying

$$\|AT(t)\| \leq \frac{C}{t}, \qquad t > 0,$$

also satisfies

$$\|A^k T(t)\| \leq \left(\frac{Ck}{t}\right)^k, \qquad t > 0,$$

and use this to conclude that T can be extended to an analytic function via

$$T(z) := \sum_{k=0}^{\infty} \frac{(z-t)^k}{k!} \frac{d^k}{dt^k} T(t), \qquad |z-t| < \frac{t}{\mathrm{e}C}.$$

Show that this extension still satisfies the semigroup property. (Hint: Problem 11.20.)

58. Let X be a topological space. A function $f : X \to \bar{\mathbb{R}}$ is called lower semicontinuous if $f^{-1}((a,\infty])$ is open for every $a \in \mathbb{R}$. Show that a lower semicontinuous is sequentially lower semicontinuous and the converse holds if X is a metric space.

59. Show that $F : M \to \overline{\mathbb{R}}$ is quasiconvex if and only if the sublevel sets $F^{-1}((-\infty, a])$ are convex for every $a \in \mathbb{R}$.

60. Let $U \subseteq \mathbb{R}^n$ be a bounded domain with a C^1 boundary. Let $\tilde{L}$ be an elliptic operator in divergence form with $A, c \in L^\infty$ and $b = 0$, $c \geq 0$. Establish existence of weak solutions in $H^1_{\mathbb{R}}(U)$ for

$$\bar{L}u = f, \qquad u\big|_{\partial U} = g.$$

61. Extend Example 13.9 to the case

$$F(u) := \frac{1}{2}\int_U |\nabla u|^2 d^n x + \int_U V(x)|u|^2 d^n x, \qquad u \in H^1_0(U,\mathbb{R}),$$

where $V \in L^q(U)$ is nonnegative with $q > \frac{n}{2}$ and $n \geq 2$.(Hint: Rellich–Kondrachov theorem.)