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MASTERARBEIT

Titel der Masterarbeit

Dispersive Estimates for One-Dimensional Schrödinger and Jacobi Operators in the Resonant Case

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angestrebter akademischer Grad
Master of Science (MSc)

Wien, im Dezember 2014

Studienkennzahl lt. Studienblatt:	A 066 821
Studienrichtung lt. Studienblatt:	Masterstudium Mathematik
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Abstract

The aim of this thesis is to obtain dispersive estimates with integrable time decay for one-dimensional continuous Schrödinger and Jacobi operators in the resonant case. In the continuous case we are able to improve previous results by weakening the assumptions on the potential, in the discrete case such an estimate hasn't been established yet.

Zusammenfassung

Das Ziel der vorliegenden Arbeit ist es, dispersive Abschätzungen mit integrierbarem Zeitabfall für kontinuierliche eindimensionale Schrödinger- und für Jakobioperatoren im Resonanzfall herzuleiten. Im kontinuierlichen Fall können wir frühere Resultate aus diesem Bereich verbessern, indem wir die Anforderungen an das Potential abschwächen, im diskreten Fall ist eine derartige Abschätzung neuartig.

Contents

0. Introduction

The purpose of this thesis is twofold: On the one hand we are interested in establishing dispersive decay estimates with integrable time decay of order $t^{-\frac{3}{2}}$ for the one-dimensional Schrödinger equation

$$\mathrm{i}\dot{\psi}(x,t) = H\psi(x,t), \quad H = -\frac{d^2}{dx^2} + V(x), \quad (x,t) \in \mathbb{R}^2, \tag{0.1}$$

with (at least) integrable potential V. We are only interested to do this in the resonant case, which means that we have a bounded solution ψ of $H\psi = 0$. We work with the spaces $L^p_\sigma = L^p_\sigma(\mathbb{R})$, $\sigma \in \mathbb{R}$, associated with the norm

$$\|\psi\|_{L^p_\sigma} = \begin{cases} \left(\int_{\mathbb{R}} (1+|x|)^{p\sigma} |\psi(x)|^p dx\right)^{1/p}, & 1 \le p < \infty, \\ \sup_{x\in\mathbb{R}} (1+|x|)^\sigma |\psi(x)|, & p = \infty, \end{cases}$$

and are able to improve an earlier result in [6] (where such an integrable time decay in the resonant case was established for $V \in L^1_4(\mathbb{R})$) by showing that $V \in L^1_3(\mathbb{R})$ is sufficient. Our calculations mostly rely on the novel fact introduced in [4], that the reflection and transmission coefficient have integrable Fourier transform, and on a simple and useful application of the van der Corput lemma, which is also taken from [4]. We use and extend the techniques mentioned in [4] appropriately. On the other hand, we prove a similar result in the discrete case, where we work with the spaces

$$\|u\|_{\ell^p_\sigma} = \begin{cases} \left(\sum_{n\in\mathbb{Z}} (1+|n|)^{p\sigma} |u(n)|\right)^{1/p}, & p \in [1,\infty), \\ \sup_{n\in\mathbb{Z}} (1+|n|)^\sigma |u(n)|, & p = \infty. \end{cases}$$

and the equation

$$\mathrm{i}\dot{u}(n,t) = a(n-1)u(n-1,t) + b(n)u(n,t) + a(n)u(n+1,t) = \widetilde{H}u(n,t), \quad (n,t) \in \mathbb{Z}\times\mathbb{R} \tag{0.2}$$

satisfying $a(n) \to 1/2$, $b(n) \to 0$ such that

$$\left|a(n) - \frac{1}{2}\right| + |b(n)| \tag{0.3}$$

is contained in $\ell^1_\sigma(\mathbb{Z})$ for some $0 \le \sigma \le 3$. The operator $\widetilde{H}$ is called the Jacobi operator. The definition of resonance is similar to the continuous case, namely $\hat{z}$ is called a resonance point, if there exists a bounded solution $\widetilde{\psi}$ of $\widetilde{H}\widetilde{\psi} = \hat{z}\widetilde{\psi}$; we have two possible points $\hat{z} = \pm 1$ of resonance though. Our main result now states, that if resonance occurs, we obtain a time decay of order $t^{-\frac{4}{3}}$, if (0.3) is an element of $\ell^1_3(\mathbb{Z})$. Our proof relies on a similar result regarding the reflection and transmission coefficients proved in [3], which states that these expressions have summable Fourier series, and again on an application of the van der Corput lemma, which is also contained in [3]. There are a lot of similarities, but also some differences between the approaches on the proofs of our two main results, to point them out the thesis has the following structure:

Section 1 deals with the Spectral Theorem of self-adjoint operators and, as an application, Stone's formula, since they are essential tools in computing our estimates. We sketch the process of constructing functions and integrals of operators, without giving detailed proofs. This section is a summary of [17, Chap. 3 and Sec. 4.1].

In **Section 2** we recall the most important facts of the Scattering theory for H,

which we need in the next section. It also contains a summary and an extension of the proof of the novel result about the scattering coefficients in [4], which we mentioned before. The results mentioned in this section are taken from [2], [13] and [20], where the detailed proofs and computations can be found.
The purpose of **Section 3** is to reach our main goal in the continuous case. We use and extend computations from [4] and [6] to obtain the desired time decay.
Section 4 gives an overview of the scattering theory for $\tilde{H}$. The structure is more or less the same as in Section 2. Again we also summarize and extend the proof of the mentioned result about the reflection and transmission coefficients taken from [3] and [5]. The material of this section is mostly taken from [16] and [18], but we use a slightly different notation.
Our final **Section 5** leads us to the desired integrable time decay in the discrete case. The structure is similar to Section 3, but sometimes the computations are a little bit different. We mostly use and extend the methods from [3] and [5].
We just want to mention, that dispersive estimates of our type play an important role in proving asymptotic stability of solitons in the continuous [1, 11] and discrete nonlinear equations [9, 10, 15].

Thanks

I want to thank my advisors Gerald Teschl and Iryna Egorova for their professional and friendly support throughout the whole writing process of this thesis. They were always there when I needed help and never gave me the impression that any of my questions were trivial. Of course I also want to thank my family for their constant support and love. Without them, this all wouldn't have been possible. Last but not least I am greatly indebted my friends and colleagues for their support, friendship and for providing an excellent counterbalance to studying, whenever I needed it.

1. The Spectral Theorem and Applications

This section is devoted to the spectral theorem of unbounded self-adjoint operators and, as an application, Stone's formula. These tools enable us to define functions of unbounded self-adjoint operators and are crucial if we want to understand, e.g., the time evolution of Schrödinger-type-operators. We follow [17, Chap. 3 and Sect. 4.1], where all the proofs and technical details can be found. For the measure-theoretical background we also refer to [17, Appendix].
As a warm-up let's recall some basic functional-analytic definitions: By $\mathfrak{H}$ we denote a complex valued and separable Hilbert-space with inner product $\langle .,. \rangle$ and norm $\| \,.\, \|$. For a densely defined linear operator $A : \mathfrak{D}(A) \to \mathfrak{H}$, the adjoint A^* of A is defined by

$$\mathfrak{D}(A^*) = \left\{ \psi \in \mathfrak{H} \,|\, \exists \widetilde{\psi} \in \mathfrak{H} : \langle \psi, A\varphi \rangle = \left\langle \widetilde{\psi}, \varphi \right\rangle \forall \varphi \in \mathfrak{D}(A) \right\}$$
$$A^*\psi = \widetilde{\psi}.$$

If we have $A = A^*$, that is $D(A) = D(A^*)$ and $A\psi = A^*\psi$ for all $\psi \in \mathfrak{D}(A)$, we call A self-adjoint. A is called normal, if $D(A) = D(A^*)$ and $\|A\psi\| = \|A^*\psi\|$ and it's called closed, if $\psi_n \to \psi$ and $A\psi_n \to \varphi$ imply $A\psi = \varphi$. Furthermore, for a closed operator A, the resolvent set of A is defined by

$$\rho(A) := \left\{ z \in \mathbb{C} \,|\, (A - z)^{-1} \in \mathfrak{L}(\mathfrak{H}) \right\},$$

where $\mathfrak{L}(\mathfrak{H})$ denotes the set of bounded linear operators on $\mathfrak{H}$. The function

$$R_A : \rho(A) \to \mathfrak{L}(\mathfrak{H})$$
$$z \mapsto (A - z)^{-1}$$

is called the resolvent of A. The complement of the resolvent set is called the spectrum $\sigma(A)$. Now we start our glimpse at the spectral theorem with an important definition: A projection valued measure P is a map

$$P : \mathfrak{B} \to \mathfrak{L}(\mathfrak{H}), \quad \Omega \mapsto P(\Omega)$$

from the Borel σ-Algebra $\mathfrak{B}$ of $\mathbb{R}$ to the set of orthogonal projections, that is $P(\Omega) = P(\Omega)^2$ and $P(\Omega)^* = P(\Omega)$, such that the following properties hold:

(i) $P(\mathbb{R}) = \mathbb{I}$
(ii) If $\Omega = \bigcup_n \Omega_n$ such that the sets Ω_n are pairwise disjoint, then $\sum_n P(\Omega_n)\psi = P(\Omega)\psi$ for every $\psi \in \mathfrak{H}$.

For any $\psi \in \mathfrak{H}$ we get a finite Borel measure $\mu_\psi(\Omega) = \langle \psi, P(\Omega)\psi \rangle$. The polarization-identity leads us to the complex-valued measures $\mu_{\varphi,\psi} = \langle \varphi, P(\Omega)\psi \rangle$. As our aim is to define $f(A)$ for a fairly large class of functions f, we proceed step by step and therefore start with simple functions $f = \sum_{j=1}^n \alpha_j \chi_{\Omega_j}$, where $\Omega_j = f^{-1}(\alpha_j)$. In this case we define $P(f) = \sum_{j=1}^n \alpha_j P(\Omega_j)$. Then $\langle \varphi, P(\Omega)\psi \rangle = \sum_j \alpha_j \mu_{\varphi,\psi}(\Omega_j)$ shows $\langle \varphi, P(\Omega)\psi \rangle = \int_{\mathbb{R}} f(\lambda) d\mu_{\varphi,\psi}(\lambda)$, so by linearity of the integral, we can deduce that P is a linear map from the set of simple functions into $\mathfrak{L}(\mathfrak{H})$. Furthermore we have

$$\|P(f)\psi\|^2 = \int_{\mathbb{R}} |f(\lambda)|^2 d\mu_\psi(\lambda) \tag{1.1}$$

and after equipping the set of simple functions with the sup-norm, we get $\|P(f)\psi\| \leq \|f\|_\infty \|\psi\|$. The simple functions are dense in the Banach-space of bounded Borel measurable functions $B(\mathbb{R})$, therefore we can find a unique extension of P to a bounded linear operator $P : B(\mathbb{R}) \to \mathfrak{L}(\mathfrak{H})$. As a consequence of (1.1) and the dominated convergence theorem we also get the following lemma, which turns out to be useful later on:

Lemma 1.1. [17, Theorem 3.1.] *If $f_n(x) \to f(x)$ pointwise for $f, f_n \in B(\mathbb{R})$, and $\sup_{\lambda\in\mathbb{R}} |f_n(\lambda)|$ is bounded, then $P(f_n) \to P(f)$ strongly.*

Our next step is to extend this process to unbounded Borel functions f. To do so we need a suitable domain first. Comparing with (1.1), the obvious choice is $\mathfrak{D}_f = \left\{\psi \in \mathfrak{H} \,|\, \int_{\mathbb{R}} |f(\lambda)|^2 d\mu_\psi(\lambda) < \infty\right\}$. One can show that this set is a densely defined linear subspace of $\mathfrak{H}$. Moreover, for every $\psi \in \mathfrak{D}_f$, the sequence of bounded Borel functions $f_n = \chi_{\Omega_n} f$, with $\Omega_n = \{\lambda \,|\, |f(\lambda)| \le n\}$, is Cauchy converging to f in $L^2(\mathbb{R}, d\mu_\psi)$. Thus, by (1.1), the vectors $\psi_n = P(f_n)\psi$ now form a Cauchy sequence in $\mathfrak{H}$ and therefore enable us to define $P(f)\psi$ as $\lim_{n\to\infty} P(f_n)\psi$. Moreover the following theorem holds:

Theorem 1.2. [17, Theorem 3.2.] *For any Borel function f, the operator $P(f)$, with $\mathfrak{D}(P(f)) = \mathfrak{D}_f$ is normal and satisfies $\|P(f)\psi\|^2 = \int_{\mathbb{R}} |f(\lambda)|^2 d\mu_\psi(\lambda)$ and $\langle\psi, P(f)\psi\rangle = \int_{\mathbb{R}} f(\lambda) d\mu_\psi(\lambda)$. Furthermore we obtain $P(f)^* = P(f^*)$ and for any Borel function g and $\alpha, \beta \in \mathbb{C}$, the following properties hold:*

$$\alpha P(f) + \beta P(g) \subseteq P(\alpha f + \beta g), \quad \mathfrak{D}(\alpha P(f) + \beta P(g)) = \mathfrak{D}_{|f|+|g|}$$

and

$$P(f)P(g) \subseteq P(fg), \quad \mathfrak{D}(P(f)P(g)) = \mathfrak{D}_g \cap \mathfrak{D}_{fg}.$$

Next we want to look closer at similarities between the operators $P(f)$ in $\mathfrak{H}$ and the multiplication operator with f in $L^2(\mathbb{R}, d\mu_\psi)$. Therefore we need some further definitions. First of all an operator $U : \mathfrak{H} \to \mathfrak{H}$ is said to be unitary if it's bijective and norm-preserving. The operators A in $\mathfrak{H}$ and $\widetilde{A}$ in $\widetilde{\mathfrak{H}}$ are said to be unitarily equivalent, if $UA = \widetilde{A}U$ and $U\mathfrak{D}(A) = \mathfrak{D}(\widetilde{A})$ for U unitary. We set

$$\mathfrak{H}_\psi = \left\{P(g)\psi \,|\, g \in L^2(\mathbb{R}, d\mu_\psi)\right\}.$$

Moreover we call $\{\psi_j\}_{j\in J}$(with some index set J) a set of spectral vectors, if $\|\psi_j\| = 1$ and $\mathfrak{H}_{\psi_i} \perp \mathfrak{H}_{\psi_j}$ for all $i \neq j$. We call a set of spectral vectors a spectral basis, if $\bigoplus_j \mathfrak{H}_{\psi_j} = \mathfrak{H}$. Then the following theorem holds.

Theorem 1.3. [17, Lemma 3.4.] *For every projection-valued measure P, there is an(at most countable) spectral basis $\{\psi_n\}$ such that*

$$\mathfrak{H} = \bigoplus_n \mathfrak{H}_{\psi_n}$$

and a corresponding unitary operator

$$U = \bigoplus_n U_{\psi_n} : \mathfrak{H} \to \bigoplus_n L^2(\mathbb{R}, d\mu_{\psi_n})$$

such that for any Borel function f,

$$UP(f) = fU \quad U\mathfrak{D}_f = \mathfrak{D}(f).$$

Our previous considerations especially show that for every projection-valued measure P there is a self-adjoint operator $A = P(\lambda)$. To finally obtain the spectral theorem it remains to prove the inverse statement, that is we need to establish the existence of a projection-valued measure P_A for every self-adjoint operator A. To do this we need the resolvent $R_A(z)$ and write $F_\psi(z) = \langle R_A(z)\psi, \psi\rangle$. We want to show that there is a unique measure μ_ψ, such that $F_\psi(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z} d\mu_\psi(\lambda)$ and call F_ψ the Borel-transform of the measure μ_ψ in this case. One can show that F_ψ is a holomorphic function, which maps the upper half-plane into itself(a so-called Herglotz or Nevalinna function) and satisfies the estimate $|F_\psi(z)| \leq \frac{\|\psi\|^2}{\mathrm{Im}(z)}$. Then it's guaranteed that $F_\psi(z)$ is the Borel-transform of a unique measure μ_ψ, which is given by the Stieltjes inversion formula

$$\mu_\psi(\lambda) = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{-\infty}^{\lambda+\delta}\mathrm{Im}(F_\psi(t+\mathrm{i}\varepsilon))dt.$$

By polarization, we can construct a corresponding complex measure $\mu_{\varphi,\psi}$ for every $\phi, \psi \in \mathfrak{H}$, such that $\langle \varphi, R_A(z)\psi\rangle = \int_{\mathbb{R}} \frac{1}{\lambda - z} d\mu_{\varphi,\psi}(\lambda)$. Now it's natural to define a family of operators via the sesquilinear forms

$$s_\Omega(\varphi, \psi) = \int_{\mathbb{R}} \chi_\Omega d\mu_{\varphi,\psi}(\lambda),$$

which satisfy $|s_\Omega(\varphi,\psi)| \leq \|\varphi\|\,\|\psi\|$. Therefore we can deduce that there is a family of nonnegative ($0 \leq \langle\psi, P_A(\Omega)\psi\rangle \leq \|\psi\|^2$) and hence self-adjoint operators $P_A(\Omega)$, which satisfy

$$\langle\varphi, P_A(\Omega)\psi\rangle = \int_{\mathbb{R}} \chi_\Omega d\mu_{\varphi,\psi}(\lambda).$$

A longer calculation shows that the family $P_A(\Omega)$ is indeed a projection valued measure. Thus we finally arrive at the spectral theorem:

Theorem 1.4 (Spectral Theorem)**.** [17, Theorem 3.6.] *To every self-adjoint operator A there corresponds a unique projection valued measure P_A, such that $A = P_A(\lambda)$.*

Now we try to characterize the spectrum of A using the associated projectors:

Theorem 1.5. [17, Theorem 3.7., Corollary 3.8.] *The spectrum of A is given by*

$$\sigma(A) = \{\lambda \in \mathbb{R} \,|\, P_A((\lambda-\varepsilon, \lambda+\varepsilon)) \neq 0 \;\forall\varepsilon > 0\}.$$

Moreover we have $P_A(\sigma(A)) = \mathbb{I}$ and $P_A(\mathbb{R}\cap\rho(A)) = 0$.

From now on we write $f(A)$ instead of $P_A(f)$. Our next goal is to have a closer look at multiplication operators on $L^2(\mathbb{R}, d\mu)$, where μ is a finite Borel measure, since they enable us to get a better understanding of arbitrary self-adjoint operators A by Theorem 1.3. We call

$$\sigma(\mu) = \{\lambda \in \mathbb{R} \,|\, \mu((\lambda-\varepsilon, \lambda+\varepsilon)) > 0 \;\forall\varepsilon > 0\}$$

the spectrum of μ. Similar calculations as for Theorem 1.5 show that the spectrum $\sigma(\mu)$ is a support for μ, that is $\mu(\mathbb{R}\backslash\sigma(\mu)) = 0$. Closely linked with the measure μ is the operator

$$Af(\lambda) = \lambda f(\lambda),\;\; \mathfrak{D}(A) = \left\{f \in L^2(\mathbb{R}, d\mu) \,|\, \lambda f(\lambda) \in L^2(\mathbb{R}, d\mu)\right\}.$$

Again by Theorem 1.5, we have $\sigma(A) = \sigma(\mu)$. Next we recall some measure-theoretical facts: The unique decomposition of μ with respect to the Lebesgue measure is given by

$$d\mu = d\mu_{ac} + d\mu_s,$$

where μ_{ac} is absolutely continuous with respect to the Lebesgue-measure(i.e. $\mu_{ac}(B) = 0$ for every set B with Lebesgue measure zero), and μ_s is singular with respect to the Lebesgue-measure(i.e., there is a Lebesgue null set B, such that $\mu(\mathbb{R}\backslash B) = 0$). Furthermore μ_s can be written as a sum of a (singularly) continuous and a pure point part,

$$d\mu_s = d\mu_{sc} + d\mu_{pp},$$

where μ_{sc} is continuous on $\mathbb{R}$ and μ_{pp} is a step function. As the measures $d\mu_{ac}$, $d\mu_{sc}$ and $d\mu_{pp}$ are mutually singular, their supports M_{ac}, M_{sc} and M_{pp} are mutually disjoint. It's important to observe that these sets are not unique. Consequently we can choose them such that M_{pp} is the set of all jumps of $\mu(\lambda)$ and such that M_{sc} is a Lebesgue null set. Next we define the corresponding projectors $P^{ac} = \chi_{M_{ac}}(A)$, $P^{sc} = \chi_{M_{sc}}(A)$, and $P^{pp} = \chi_{M_{pp}}(A)$, which satisfy $P^{ac} + P^{sc} + P^{pp} = \mathbb{I}$. Thus both our Hilbert space $L^2(\mathbb{R}, d\mu)$ and our operator A can be written as direct sums:

$$L^2(\mathbb{R}, d\mu) = L^2(\mathbb{R}, d\mu_{ac}) \oplus L^2(\mathbb{R}, d\mu_{sc}) \oplus L^2(\mathbb{R}, d\mu_{pp})$$

and

$$A = (AP^{ac}) \oplus (AP^{sc}) \oplus (AP^{pp}).$$

Their spectra, $\sigma_{ac}(A) = \sigma(\mu_{ac})$, $\sigma_{sc}(A) = \sigma(\mu_{sc})$ and $\sigma_{pp}(A) = \sigma(\mu_{pp})$, are called the absolutely continuous, singularly continuous, and pure point spectrum of A, respectively.

Using Theorem 1.3, these results can be transferred to arbitrary self-adjoint operators A as well. Therefore we need a spectral measure, which gives us all the information from all measures in a spectral basis. This is the case if we have a vector ψ, such that for every $\varphi \in \mathfrak{H}$ its spectral measure μ_φ is absolutely continuous with respect to μ_ψ. Then we call ψ a maximal spectral vector of A and μ_ψ a maximal spectral measure of A. It can be shown that such a maximal spectral vector always exists for a self-adjoint operator A. A set $\{\psi_j\}$ of spectral vectors is called ordered, if ψ_k is a maximal spectral vector for A restricted to $\left(\bigoplus_j^{k-1} \mathfrak{H}_{\psi_j}\right)^\perp$. We can even show that for every self-adjoint operator there is an ordered spectral basis. Next we define

$$\begin{aligned} \mathfrak{H}_{ac} &= \{\psi \in \mathfrak{H} \,|\, \mu_\psi \text{ is absolutely continuous}\} \\ \mathfrak{H}_{sc} &= \{\psi \in \mathfrak{H} \,|\, \mu_\psi \text{ is singularly continuous}\} \\ \mathfrak{H}_{pp} &= \{\psi \in \mathfrak{H} \,|\, \mu_\psi \text{ is pure point}\}. \end{aligned}$$

Then we get a similar operator splitting as before:

Theorem 1.6. [17, Lemma 3.18.] *We have*

$$\mathfrak{H} = \mathfrak{H}_{ac} \oplus \mathfrak{H}_{sc} \oplus \mathfrak{H}_{pp}.$$

There are Borel sets M_{xx} such that the projector onto $\mathfrak{H}_{xx}$ is given by $P^{xx} = \chi_{M_{xx}}$, $xx \in \{ac,sc,pp\}$. For the sets M_{xx} one can choose the corresponding supports of some maximal spectral measure μ.

Finally the definition of the absolutely continuous, singularly continuous, and pure point spectrum of A is given by

$$\sigma_{ac}(A) = \sigma(A|_{\mathfrak{H}_{ac}}),\ \sigma_{sc}(A) = \sigma(A|_{\mathfrak{H}_{sc}}),\ \sigma_{pp}(A) = \sigma(A|_{\mathfrak{H}_{pp}}).$$

Moreover, if μ is a maximal spectral measure, the corresponding spectra of A and μ coincide.

The remaining part of this section is now devoted to Stone's formula, as an useful application of the Spectral theorem. Therefore we first have a look at integration in Banach spaces. We look at mappings $f : I \to X$, where $I = [t_0, t_1]$ is a compact interval, X a Banach space and μ a (σ)-finite Borel measure. f is called simple if its image is finite, $f(I) = \{x_i\}_{i=1}^n$, and if each inverse image $f^{-1}(x_i)$, $1 \le i \le n$, is measurable. The set of simple functions $S(I, X)$ is a linear space that can be equipped with the sup norm $\|f\|_\infty = \sup_{t\in I} \|f(t)\|$. We call the corresponding Banach space, which we get after completion, the set of regular functions $R(I, X)$. Furthermore we can easily deduce that $C(I, X) \subset R(I, X)$. For $f \in S(I, X)$ it is now possible to define a linear map $\int : S(I, X) \to X$ by

$$\int_I f(\lambda) d\mu(\lambda) = \sum_{i=1}^n x_i \mu(f^{-1}(x_i)).$$

This map has the property, that

$$\left\| \int_I f(\lambda) d\mu(\lambda) \right\| \le \|f\|_\infty \mu(I),$$

and hence can be extended uniquely to a linear map $\int : R(I, X) \to X$ with the same norm. We even have

$$\left\| \int_I f(\lambda) d\mu(\lambda) \right\| \le \int_I \|f(\lambda)\| \, d\mu(\lambda), \tag{1.2}$$

which is valid for simple functions and therefore for all $f \in R(I, X)$ by continuity. Furthermore, if $A(t) \in R(I, \mathfrak{L}(\mathfrak{H}))$, then

$$\left(\int_I A(\lambda) d\mu(\lambda) \right) \psi = \int_I (A(\lambda)\psi) d\mu(\lambda).$$

If $I = \mathbb{R}$, we call $f : I \to X$ integrable, if $f \in R([-r, r], X)$ for all $r > 0$ and if $\|f(t)\|$ is integrable with respect to μ. Then we define

$$\int_{\mathbb{R}} f(\lambda) d\mu(\lambda) = \lim_{r\to\infty} \int_{[-r,r]} f(\lambda) d\mu(\lambda),$$

and (1.2) continues to hold. For us, the following theorem, which is basically a consequence of Theorem 1.4 and Fubini's theorem, is of interest:

Theorem 1.7. [17, Lemma 4.1.] *Suppose $f : I \times \mathbb{R} \to \mathbb{C}$ is a bounded μ-measurable function and set $F(x) = \int_I f(\lambda, x) d\mu(\lambda)$. Let A be self-adjoint. Then $f(\lambda, A) \in R(I, \mathfrak{L}(\mathfrak{H}))$ and $F(A) = \int_I f(\lambda, A) d\mu(\lambda)$, respectively, $F(A)\psi = \int_I f(t, A)\psi d\mu(\lambda)$.*

Now, finally we are ready to formulate and prove Stone's formula in the form we need it:

Theorem 1.8 (Stone)**.** [17, Problem 4.3.] *Suppose $f \in C(\mathbb{R})$ is bounded and A self-adjoint. Then*

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} f(\lambda)(R_A(\lambda + \mathrm{i}\varepsilon) - R_A(\lambda - \mathrm{i}\varepsilon)) d\lambda \xrightarrow{s} f(A)$$

strongly. Furthermore we have

$$\frac{1}{2\pi \mathrm{i}} \int_{\lambda_1}^{\lambda_2} f(\lambda)(R_A(\lambda + \mathrm{i}\varepsilon) - R_A(\lambda - \mathrm{i}\varepsilon)) d\lambda \xrightarrow{s} \frac{1}{2}(P_A([\lambda_1, \lambda_2]) + P_A((\lambda_1, \lambda_2)))f(A).$$

Proof. For the first claim we observe that

$$\frac{1}{x-\lambda-\mathrm{i}\varepsilon}-\frac{1}{x-\lambda+\mathrm{i}\varepsilon}=\frac{2\mathrm{i}\varepsilon}{(\lambda-x)^2+\varepsilon^2}.$$

Now we consider the integral $\frac{1}{\pi}\int_{\mathbb{R}} f(\lambda)\frac{\varepsilon}{(\lambda-x)^2+\varepsilon^2}d\lambda$ and use the substitution $u=\frac{\lambda-x}{\varepsilon}$ to rewrite this expression as

$$\frac{1}{\pi}\int_{\mathbb{R}} f(x+\varepsilon u)\frac{1}{u^2+1}du=\frac{1}{\pi}\int_{\mathbb{R}}(f(x+\varepsilon u)-f(x))\frac{1}{u^2+1}du+\frac{1}{\pi}\int_{\mathbb{R}} f(x)\frac{1}{u^2+1}du.$$

To treat the first summand let's write $g_\varepsilon(u)=(f(x+\varepsilon u)-f(x))\frac{1}{u^2+1}$ and use the boundedness of f to obtain that for every $\varepsilon>0$, $|g_\varepsilon(u)|$ is bounded above by $\frac{C}{u^2+1}$, which is integrable. Therefore we can apply dominated convergence and we use the continuity of f to deduce that $\frac{1}{\pi}\int_{\mathbb{R}} g_\varepsilon(\lambda)d\lambda$ converges to zero as $\varepsilon\to 0$. The second summand can now be written as $f(x)$ and thus we have shown that

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}} f(\lambda)(\frac{1}{x-\lambda-\mathrm{i}\varepsilon}-\frac{1}{x-\lambda+\mathrm{i}\varepsilon})d\lambda\to f(x).$$

Combining Lemma 1.1 and Theorem 1.7 proves the first claim.
The second claim follows from the first, if f vanishes at the boundary points. Thus it remains to consider the cases $f(\lambda)=1$ and $f(\lambda)=\lambda$, which can be verified by a straightforward calculation. □

2. Continuous One-Dimensional Schrödinger Operators and their Scattering Theory

In this section we consider the scattering theory of H, defined in (0.1). In [17, Sect. 9.7] it is proved, that for $V \in L^1_1$ the operator H is self-adjoint on the domain

$$\mathfrak{D}(H) = \left\{ \psi \in L^2(\mathbb{R}) \,|\, \psi, \psi' \text{ are locally absolutely continuous, } H\psi \in L^2(\mathbb{R}) \right\},$$

has a purely absolutely continuous spectrum on $[0, \infty)$ and a finite number of eigenvalues in $(-\infty, 0)$. Our next step is to have a look at the scattering theory of H. To do so we start with a theorem about solutions of certain integral equations. They are solved by the method of successive approximation. Since this tool also plays an important role in our computations later on, we explain it in all detail. For further background see [2, 13].

Theorem 2.1 (Volterra Integral equations). *Suppose $f \in L^\infty(-\infty, x]$ for every $x \in \mathbb{R}$, and for $y \leq x$ let $K(x,y)$ satisfy the conditions*

(i) K is measurable and $|K(x,y)| \leq K_1(x,y)$, where $K_1(x,y)$ is non-decreasing in x for each $y \leq x$.

(ii) For each fixed $x \in \mathbb{R}$, $K_1(x,y)$ is integrable with respect to y on the interval $(-\infty, x]$.

Then

$$g(x) = f(x) + \int_{-\infty}^{x} K(x,y) g(y) dy$$

has a unique solution g which satisfies the estimate

$$|g(x) - f(x)| \leq \sup_{y \in (-\infty, x]} |f(y)| \cdot \int_{-\infty}^{x} K_1(x,y) dy \cdot \exp\left(\int_{-\infty}^{x} K_1(x,y) dy \right)$$

Proof. Set $g_0(x) = f(x)$, $g_{n+1}(x) = \int_{-\infty}^{x} K(x,y) g_n(y) dy$. By induction we observe that $g_n(x)$ is well-defined for each $n \in \mathbb{N}$, using the assumptions on f and K. Furthermore

$$\sum_{k=0}^{n} g_k(x) = f(x) + \int_{-\infty}^{x} K(x,y) \sum_{k=0}^{n-1} g_k(y) dy.$$

To establish existence we need that the series $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on every compact interval in $\mathbb{R}$. We therefore show

$$|g_n(x)| \leq \sup_{y \in (-\infty, x]} |f(y)| \cdot \frac{1}{n!} \left(\int_{-\infty}^{x} K_1(x,y) dy \right)^n \tag{2.1}$$

inductively first. For $n = 0$ it's obvious. Now assume that it's true for some $n \in \mathbb{N}$. Then we get

$$\begin{aligned}
|g_{n+1}(x)| &\leq \int_{-\infty}^{x} K_1(x,y) |g_n(y)| dy \\
&\leq \int_{-\infty}^{x} K_1(x,y) \sup_{z \in (-\infty, y]} |f(z)| \cdot \frac{1}{n!} \left(\int_{-\infty}^{y} K_1(y,z) dz \right)^n dy \\
&\leq \sup_{y \in (-\infty, x]} |f(y)| \int_{-\infty}^{x} K_1(x,y) \cdot \frac{1}{n!} \left(\int_{-\infty}^{y} K_1(x,z) dz \right)^n dy \\
&= \sup_{y \in (-\infty, x]} |f(y)| \frac{1}{(n+1)!} \left(\int_{-\infty}^{x} K_1(x,y) dy \right)^{n+1},
\end{aligned}$$

where the last equality is a consequence of the substitution $u = \int_{-\infty}^{y} K_1(x,z)dz$. Therefore (2.1) implies

$$\sum_{n=1}^{\infty} |g_n(x)| \le \sup_{y\in(-\infty,x]} |f(y)| \cdot \sum_{n=1}^{\infty} \frac{1}{n!} \left(\int_{-\infty}^{x} K_1(x,y)dy \right)^n$$
$$= \sup_{y\in(-\infty,x]} |f(y)| \int_{-\infty}^{x} K_1(x,y)dy \cdot \exp\left(\int_{-\infty}^{x} K_1(x,y)dy \right).$$

Thus we have established the absolute convergence of the series on every compact interval in $\mathbb{R}$ and therefore, the existence of the solution of the integral equation and the desired estimate follow immediately. Our next aim is to show uniqueness. Let therefore g and $\widetilde{g}$ be two solutions of the integral equation and set $h = g - \widetilde{g}$. Then $h(x) = \int_{-\infty}^{x} K(x,y)h(y)dy$. Using (2.1) gives us

$$|h(x)| \le \sup_{y\in(-\infty,x]} |h(y)| \cdot \frac{1}{n!} \left(\int_{-\infty}^{x} K_1(x,y)dy \right)^n,$$

which converges to 0 as $n \to \infty$.

□

Now we apply this result to establish existence and uniqueness of solutions f of the equation $Hf = k^2 f$ for some complex number k. We also mention asymptotics of these solutions and their derivatives. For notational simplicity we introduce the functions

$$\gamma_\pm(x) = \int_x^{\pm\infty} (y-x)|V(y)|dy, \text{ and } \quad \eta_\pm(x) = \pm \int_x^{\pm\infty} |V(y)|dy. \tag{2.2}$$

Theorem 2.2. *Let $V \in L^1_1(\mathbb{R})$. Then for each k with $0 \le \mathrm{Im}(k)$ the integral equations*

$$f_\pm(x,k) = \mathrm{e}^{\pm \mathrm{i}kx} - \int_x^{\pm\infty} \frac{\sin(k(x-y))}{k} V(y) f_\pm(y,k)dy \tag{2.3}$$

have unique solutions defined everywhere on $\mathbb{R}$, which solve the Schrödinger-equation

$$H\psi = k^2\psi. \tag{2.4}$$

For each x the functions $f_\pm(x,k)$, $\frac{\partial}{\partial x} f_\pm(x,k) = f'_\pm(x,k)$ are analytic in the upper half-plane $\mathrm{Im}(k) > 0$ and continuous in $0 \le \mathrm{Im}(k)$. They satisfy the following estimates:

$$\left| f_\pm(x,k) - \mathrm{e}^{\pm \mathrm{i}kx} \right| \le \frac{const}{|k|} \exp\left(\frac{const}{|k|}\right) \mathrm{e}^{\mp \mathrm{Im}(k)x}, \quad k \ne 0 \tag{2.5}$$

$$|f_\pm(x,k)| \le const(1 + \max\{x,0\}) \mathrm{e}^{\mp \mathrm{Im}(k)x} \tag{2.6}$$

$$\left| f'_\pm(x,k) \right| \le const\left(\frac{1+|k|}{|k|}\right) \mathrm{e}^{\mp \mathrm{Im}(k)x}, \quad k \ne 0 \tag{2.7}$$

$$\left| f'_\pm(x,k) \right| \le const(1 + |k| + |k||x|) \mathrm{e}^{\mp \mathrm{Im}(k)x}, \tag{2.8}$$

where const denotes a constant, that is independent of x and k. If $V \in L^1_2(\mathbb{R})$, then $\frac{\partial}{\partial k} f_\pm(x,k) = \dot{f}(x,k)$ exists for $0 \le \mathrm{Im}(k)$ and is continuous there as a function of

k. Furthermore the following estimates are valid:

$$|\frac{\partial}{\partial k}\left(\mathrm{e}^{\mp \mathrm{i}kx} f_\pm(x,k)\right)| \leq const(1+x^2) \tag{2.9}$$

$$|\frac{\partial}{\partial k}\left(f_\pm(x,k)\right)| \leq const(1+x^2)\mathrm{e}^{\mp \operatorname{Im}(k)x} \tag{2.10}$$

Proof. We only sketch the proof. For details we refer for example to [20, Chapter 5, Theorem 26]. We introduce the function

$$h_\pm(x,k) = \mathrm{e}^{\mp \mathrm{i}kx} f_\pm(x,k), \tag{2.11}$$

but only consider the $-$ -case, since the $+$ -case is similar. (2.3) for f_- can be written as

$$h_-(x,k) = 1 + \int_{-\infty}^{x} \frac{\mathrm{e}^{2\mathrm{i}k(x-y)}-1}{2\mathrm{i}k} V(y) h_-(y,k) dy.$$

For the kernel $\overline{K}(x,y) = \frac{\mathrm{e}^{2\mathrm{i}k(x-y)}-1}{2\mathrm{i}k} V(y)$ we get the estimates

$$|\overline{K}(x,y)| \leq \frac{|V(y)|}{|k|}, \quad \text{i.e.,} \quad \int_{-\infty}^{x} |\overline{K}(x,y)| dy \leq \frac{\|V\|_1}{|k|}, \quad k \neq 0 \tag{2.12}$$

and

$$|\overline{K}(x,y)| \leq (x-y)|V(y)|, \quad \text{i.e.,} \quad \int_{-\infty}^{x} |\overline{K}(x,y)| dy \leq \max\{x,0\} \|V\|_1 + \|V\|_1^1 \tag{2.13}$$

for $0 \leq x - y$ and $0 \leq \operatorname{Im}(k)$. If we use (2.12) and apply Theorem 2.1, we get that for $k \neq 0$ h_- and hence f_- are uniquely determined functions, and also (2.5) follows immediately. Inserting (2.5) into

$$h'_-(x,k) = \int_{-\infty}^{x} \mathrm{e}^{2ik(x-y)} V(y) h_-(x,y) dy$$

gives us estimate (2.7). If we apply Gronwall's inequality (c.f. [18, Lemma 2.7]) to (2.13), we get

$$|h_-(x,k) - 1| \leq \gamma_-(x) \mathrm{e}^{\gamma_-(x)}, \tag{2.14}$$

which gives us existence and uniqueness also for $k = 0$. A direct calculation using the Leibnitz rule for parameter integrals shows that $f_\pm$ solves the Schrödinger equation. Since the series for m_- is locally uniformly convergent, we get analyticity of h_- and hence of f_- in $\operatorname{Im}(k) > 0$ and continuity in $0 \leq \operatorname{Im}(k)$. To establish (2.6), we first observe that $|h_-(x,k)| \leq 1 + \gamma_-(0)\mathrm{e}^{\gamma_-(0)}$ for $x < 0$. For $x > 0$ we use (2.13) to obtain

$$|h_-(x,k)| \leq K + \int_{-\infty}^{x} x(1+|y|)|V(y)| \frac{|h_-(y,k)|}{1+|y|} dy$$

for some $0 \leq K$. Setting $M(x,k) = \frac{|h_-(x,k)|}{K(1+|x|)}$ leads us to the following inequality for M:

$$M(x,k) \leq 1 + \int_{-\infty}^{x} (1+|y|)|V(y)| M(y,k) dy.$$

Inserting this inequality repeatedly into itself, we finally arrive, after some computations, at

$$M(x,k) \leq \exp\left(\int_{-\infty}^{x} (1+|y|)|V(y)| dy\right) \leq \text{const}.$$

This proves (2.6). The inequality (2.8) is now obtained in a similar way as (2.7). The remaining estimates (2.9) and (2.10) can be derived using a similar iteration process as before. □

From now on we always use the symbol $'$ as an abbreviation for the derivative with respect to the variable x, and $\cdot$ for the derivative w.r.t. k. Next we consider the functions $\widetilde{f}_\pm(x,k) = f_\pm(x,k^*)^*$, which solve the equation

$$\widetilde{f}_\pm(x,k) = \mathrm{e}^{\mp \mathrm{i}kx} - \int_x^{\pm\infty} \frac{\sin(k(x-y))}{k} V(y)\widetilde{f}_\pm(y,k)dy$$

for $\mathrm{Im}(k) \le 0$. By $W(\varphi(x,k),\psi(x,k)) = \varphi(x,k)\psi'(x,k) - \varphi'(x,k)\psi(x,k)$ we denote the usual Wronskian. A straightforward calculation, using the fact that $f_\pm(x,k)$ and $\widetilde{f}_\pm(x,k)$ solve the Schrödinger equation (2.4), shows $\frac{\partial}{\partial x}\Big(W(f_\pm(x,k),\widetilde{f}_\pm(x,k))\Big) = 0$. Thus to calculate $W(f_\pm(x,k),\widetilde{f}_\pm(x,k))$ it's sufficient to consider

$$\lim_{x\to\pm\infty} W(f_\pm(x,k),\widetilde{f}_\pm(x,k)) = \mp 2\mathrm{i}k. \tag{2.15}$$

Furthermore we use the following expressions:

$$W(k) = W(f_-(x,k), f_+(x,k)), \qquad W_\pm(k) = W(f_\mp(x,k),\widetilde{f}_\pm(x,k)).$$

If k is real valued, we obtain four solutions of (2.4), which obviously can't be linearly independent. However as a consequence of (2.15), we have that the pairs $(f_+,\widetilde{f}_+)$ and $(f_-,\widetilde{f}_-)$ are linearly independent. Since for real k we even have $\widetilde{f}_\pm(x,k) = f_\pm(x,-k)$, this finally leads, after some calculations, to the scattering relations

$$T(k)f_\pm(x,k) = R_\mp(k)f_\mp(x,k) + f_\mp(x,-k), \quad k \neq 0 \tag{2.16}$$

where T and $R_\pm$ denote the transmission and reflection coefficients, given by

$$T(k) = \frac{2\mathrm{i}k}{W(k)}, \quad R_\pm(k) = \mp\frac{W_\pm(k)}{W(k)}. \tag{2.17}$$

Some more important properties concerning these coefficients are e.g. that

$$g_1(k) = \frac{1}{T(k)} \neq 0, \quad k \in \mathbb{R}, \tag{2.18}$$

and that $T(k)$ and $R_\pm(k)$ both are continuous at $k = 0$. For the proof we again refer to [20, Proposition 27]. The continuity of $R_\pm$ was established in [8]. Moreover, by a direct calculation, one can show the following useful identities:

$$|T(k)|^2 + |R_\pm(k)|^2 = 1 \quad \text{and} \quad R_+^*(k)T(k) + T^*(k)R_-(k) = 0. \tag{2.19}$$

Finally we also need to mention, that the presence of a resonance at $k = 0$ is equivalent to the fact, that the Wronskian $W(0)$ of the two Jost solutions at this point disappears. Our next aim is to state some results about the Fourier transform of $h_\pm(x,k)$ with respect to k. Therefore we first introduce the Banach algebra $\mathcal{A}$ of Fourier transforms of integrable functions

$$\mathcal{A} = \left\{ f(k):\ f(k) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kp}\hat{f}(p)dp,\ \hat{f}(\cdot) \in L^1(\mathbb{R}) \right\} \tag{2.20}$$

with the norm $\|f\|_{\mathcal{A}} = \|\hat{f}\|_{L^1}$, and the corresponding unital Banach algebra $\mathcal{A}_1$

$$\mathcal{A}_1 = \left\{ f(k):\ f(k) = c + \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kp}\hat{g}(p)dp,\ \hat{g}(\cdot) \in L^1(\mathbb{R}),\ c \in \mathbb{C} \right\} \tag{2.21}$$

with the norm $\|f\|_{\mathcal{A}_1} = |c| + \|\hat{g}\|_{L^1}$. It's immediately clear, that $\mathcal{A}$ is a subalgebra of $\mathcal{A}_1$. The algebra $\mathcal{A}_1$ can be seen as an algebra of Fourier transforms of functions $c\delta(\cdot) + \hat{g}(\cdot)$, where δ is the Dirac delta distribution and $\hat{g} \in L^1(\mathbb{R})$. Later on we often use the important fact, that if $f \in \mathcal{A}_1 \setminus \mathcal{A}$ and $f(k) \neq 0$ for all $k \in \mathbb{R}$ then $f^{-1}(k) \in \mathcal{A}_1$ by the Wiener lemma [21]. For the Jost solutions, the following theorem is valid:

Theorem 2.3.

$$h_\pm(x,k) = 1 \pm \int_0^{\pm\infty} B_\pm(x,y) \mathrm{e}^{\pm 2\mathrm{i}ky} dy, \tag{2.22}$$

where $B_\pm(x,y)$ are real-valued and satisfy

$$|B_\pm(x,y)| \leq \mathrm{e}^{\gamma_\pm(x)} \eta_\pm(x+y), \tag{2.23}$$

$$|\frac{\partial}{\partial x} B_\pm(x,y) \pm V(x+y)| \leq 2\mathrm{e}^{\gamma_\pm(x)} \eta_\pm(x+y)\eta_\pm(x). \tag{2.24}$$

One of the major tools used to obtain the previous theorem, which also plays a key role in our remaining computations, is the following important result by Marchenko:

Theorem 2.4. *The kernels $B_\pm(x,y)$ defined by* (2.22)*, satisfy the following equation*

$$F_\pm(x+y) + B_\pm(x,y) \pm \int_0^{\pm\infty} B_\pm(x,z) F_\pm(x+y+z) dz = 0, \tag{2.25}$$

where the functions $F_\pm(x)$ given by [13, 3.5.19'] *are absolutely continuous with $F'_\pm \in L^1(\mathbb{R}_\pm)$ and*

$$|F_\pm(x)| \leq C\eta_\pm(x), \quad \pm x \geq 0, \tag{2.26}$$

with $\eta_\pm$ from (2.2)*.*

For the proofs of the previous two facts we refer to [13, §3.5] or [20, §5.2.]. In all these computations Cauchy's theorem is crucial. Moreover

$$h_\pm(x,\cdot) - 1,\ h'_\pm(x,\cdot) \in \mathcal{A}, \quad \forall x \in \mathbb{R}, \tag{2.27}$$

is an easy consequence of (2.22)–(2.24) and the $\mathcal{A}$-norms of these expressions don't depend on x, if $\pm x \geq 0$, which also follows from these estimates. Now we have a closer look at a novel result by Egorova, Kopylova, Teschl and Marchenko. We explain the techniques used there in all detail and also extend some of their results:

Theorem 2.5. [4, Theorem 2.1] *If $V \in L^1_1$, then $T(k) - 1 \in \mathcal{A}$ and $R_\pm(k) \in \mathcal{A}$.*

Proof. We only consider the resonant case, since this is the area we mainly focus on. First of all, we abbreviate $h_\pm(k) = h_\pm(0,k)$, $h'_\pm(k) = h'_\pm(0,k)$. Then (2.11) implies

$$W(k) = 2\mathrm{i}kh_+(k)h_-(k) + \tilde{W}(k), \quad \tilde{W}(k) = h_-(k)h'_+(k) - h'_-(k)h_+(k), \tag{2.28}$$

$$W_\pm(k) = h_\mp(k)h'_\pm(-k) - h_\pm(-k)h'_\mp(k). \tag{2.29}$$

Moreover, $\tilde{W}(k)$, $W_\pm(k) \in \mathcal{A}$. In the next step, we first introduce some new functions

$$\Phi_\pm(k) = h_\pm(k)h'_\pm(0) - h'_\pm(k)h_\pm(0), \tag{2.30}$$

$$K_\pm(x) = \pm\int_x^{\pm\infty} B_\pm(0,y)dy, \quad D_\pm(x) = \pm\int_x^{\pm\infty} \frac{\partial}{\partial x} B_\pm(0,y)dy, \tag{2.31}$$

where $B_\pm(x,y)$ are the transformation operators from (2.22). Integrating (2.22) formally by parts we obtain

$$\begin{aligned} h'_\pm(k) &= \pm\int_0^{\pm\infty}\frac{\partial}{\partial x}B_\pm(0,y)\mathrm{e}^{\pm 2\mathrm{i}ky}dy = D_\pm(0)+2\mathrm{i}k\int_0^{\pm\infty}D_\pm(y)\mathrm{e}^{\pm 2\mathrm{i}ky}dy\\ &= h'_\pm(0)+2\mathrm{i}k\int_0^{\pm\infty}D_\pm(y)\mathrm{e}^{\pm 2\mathrm{i}ky}dy,\\ h_\pm(k) &= h_\pm(0)+2\mathrm{i}k\int_0^{\pm\infty}K_\pm(y)\mathrm{e}^{\pm 2\mathrm{i}ky}dy. \end{aligned}$$

All the above integrals have to be understood as improper integrals. Inserting them into (2.30) gives

$$\Phi_\pm(k)=2\mathrm{i}k\Psi_\pm(k), \Psi_\pm(k)=\int_0^{\pm\infty}(D_\pm(y)h_\pm(0)-K_\pm(y)h'_\pm(0))\mathrm{e}^{\pm 2\mathrm{i}ky}dy. \quad (2.32)$$

Moreover we define

$$H_\pm(x)=D_\pm(x)h_\pm(0)-K_\pm(x)h'_\pm(0) \quad (2.33)$$

and prove the following lemma, which is an extension of [4, Lemma 2.2]:

Lemma 2.6. *If $V\in L_1^1$ then $\Psi_\pm(k)\in\mathcal{A}$. Moreover we have that $|H_\pm(x)|\le \hat{C}\eta_\pm(x)$ for some universal constant $\hat{C}>0$ and $\eta_\pm(x)$ given by* (2.2).

Proof. The proof is inspired by [8] and therefore we first take advantage of Theorem 2.4. On the one hand we differentiate (2.25) with respect to x and set $x=0$, on the other hand we set $x=0$ in (2.25) and then integrate both equations with respect to y from x to $\pm\infty$. Then (2.31) implies

$$\pm\int_x^{\pm\infty}F_\pm(y)dy+K_\pm(x)+\int_0^{\pm\infty}B_\pm(0,z)\int_x^{\pm\infty}F_\pm(y+z)dy\,dz=0$$

and

$$\begin{aligned} \mp F_\pm(x)+D_\pm(x)+\int_0^{\pm\infty}\frac{\partial}{\partial x}B_\pm(0,z)\int_x^{\pm\infty}F_\pm(y+z)dy\,dz&\\ -\int_0^{\pm\infty}B_\pm(0,z)F_\pm(x+z)dz=0.& \end{aligned}$$

Next, we apply (2.31) and the identity

$$\frac{\partial}{\partial z}\int_x^{\pm\infty}F_\pm(y+z)dy=-F_\pm(x+z).$$

Integration by parts leads to

$$\begin{aligned} &\pm(1+K_\pm(0))\int_x^{\pm\infty}F_\pm(y)dy+K_\pm(x)\mp\int_0^{\pm\infty}K_\pm(z)F_\pm(x+z)dz \quad (2.34)\\ &=K_\pm(x)\pm h_\pm(0)\int_x^{\pm\infty}F_\pm(y)dy\mp\int_0^{\pm\infty}K_\pm(z)F_\pm(x+z)dz=0 \end{aligned}$$

and

$$\begin{aligned} &\mp F_\pm(x)+D_\pm(x)\pm h'_\pm(0)\int_x^{\pm\infty}F_\pm(y)dy\mp\int_0^{\pm\infty}D_\pm(z)F_\pm(x+z)dz \quad (2.35)\\ &-\int_0^{\pm\infty}B_\pm(0,z)F_\pm(x+z)dz=0. \end{aligned}$$

Furthermore we multiply (2.34) by $h'_\pm(0)$ and (2.35) by $h_\pm(0)$ and subtracting, we get integral equations

$$H_\pm(x) \mp \int_0^{\pm\infty} H_\pm(y) F_\pm(x+y)dy = G_\pm(x), \tag{2.36}$$

where

$$G_\pm(x) = h_\pm(0)\Big(\int_0^{\pm\infty} B_\pm(0,y)F_\pm(x+y)dy \pm F_\pm(x)\Big).$$

(2.23) and (2.26) imply

$$|G_\pm(x)| \le \widetilde{C}\eta_\pm(x), \quad \pm x \ge 0. \tag{2.37}$$

As we want to obtain $H_\pm \in L^1(\mathbb{R}_\pm)$, we apply the method of successive approximations to (2.36). To this end let $N > 0$ such that $\pm C \int_{\pm N}^{\pm\infty} \eta_\pm(y)dy < 1$, where C is given by (2.26). Then we can rewrite (2.36) in the form

$$H_\pm(x) \mp \int_{\pm N}^{\pm\infty} H_\pm(y)F_\pm(x+y)dy = G_\pm(x,N), \tag{2.38}$$

where

$$G_\pm(x,N) = G_\pm(x) \pm \int_0^{\pm N} H_\pm(y)F_\pm(x+y)dy. \tag{2.39}$$

From the formulas (2.31) and the estimates (2.23)–(2.24) we deduce $H_\pm \in L^\infty(\mathbb{R}_\pm) \cap C(\mathbb{R}_\pm)$. We also have

$$|G_\pm(x,N)| \le C(N)\eta_\pm(x) \tag{2.40}$$

by the boundedness of $H_\pm$, (2.37) and monotonicity of $\eta_\pm(x)$. Now let

$$H_{\pm,0}(x) = G(x,N), \quad H_{\pm,n+1}(x) = \int_{\pm N}^{\pm\infty} H_{\pm,n}(y)F_\pm(x+y)dy.$$

We show that

$$|H_{\pm,n}(x)| \le C(N)\eta_\pm(x) \left[\pm C \int_{\pm N}^{\pm\infty} \eta_\pm(y)dy \right]^n. \tag{2.41}$$

Then we are done by Theorem 2.1, and the desired estimate for $H_\pm(x)$ also follows immediately, because

$$|H_\pm(x)| \le \ C(N)\eta_\pm(x) \sum_{n=0}^{\infty} \left[\pm C \int_{\pm N}^{\pm\infty} \eta_\pm(y)dy \right]^n = \hat{C}(N)\eta_\pm(x).$$

So it remains to verify (2.41) by induction. Equation (2.40) ensures the estimate for $n = 0$. Now assume it is true for n. Then, using (2.26) and monotonicity of $\eta_\pm(x)$, we finally get

$$\begin{aligned} |H_{\pm,n+1}(x)| &\le \pm C(N) \int_{\pm N}^{\pm\infty} \eta_\pm(y) \left[\pm C \int_{\pm N}^{\pm\infty} \eta_\pm(z)dz \right]^n |F_\pm(x+y)|dy \\ &\le \pm C(N) \left[\pm C \int_{\pm N}^{\pm\infty} \eta_\pm(z)dz \right]^n \int_{\pm N}^{\pm\infty} \eta_\pm(y) C \eta_\pm(x+y)dy \\ &\le C(N)\eta_\pm(x) \left[\pm C \int_{\pm N}^{\pm\infty} \eta_\pm(z)dz \right]^{n+1}. \end{aligned}$$

□

Now we can continue the proof of Theorem 2.5. Since the Jost solutions are linearly dependent at $k = 0$, which means that $h_+(x,0) = c\,h_-(x,0)$, we can consider two cases: (a) $h_+(0)h_-(0) \neq 0$ and (b) $h_+(0) = h_-(0) = 0$. In both cases we have $\tilde{W}(0) = W_\pm(0) = 0$ (see (2.28)–(2.29)).

Let us look at case (a) first. We have

$$\begin{aligned}\tilde{W}(k) = \tilde{W}(k) - \tilde{W}(0) &= \frac{h_-(k)}{h_+(0)}\Phi_+(k) - \frac{h_+(k)}{h_-(0)}\Phi_-(k)\\ &= 2\mathrm{i}k\Big(\frac{h_-(k)}{h_+(0)}\Psi_+(k) - \frac{h_+(k)}{h_-(0)}\Psi_-(k)\Big) = 2\mathrm{i}k\Psi(k),\end{aligned} \tag{2.42}$$

where $\Psi(k) \in \mathcal{A}$ by Lemma 2.6. Thus

$$W(k) = 2\mathrm{i}k(h_-(k)h_+(k) + \Psi(k)). \tag{2.43}$$

This representation together with (2.27) and (2.18), gives us

$$g_1(k) \in \mathcal{A}_1 \setminus \mathcal{A}. \tag{2.44}$$

Therefore,

$$T(k) = \frac{1}{g_1(k)} \in \mathcal{A}_1, \quad T(k) - 1 \in \mathcal{A}. \tag{2.45}$$

by the Wiener lemma. Analogously,

$$W_\pm(k) = \frac{h_\mp(k)}{h_\pm(0)}\Phi_\pm(-k) - \frac{h_\pm(-k)}{h_\mp(0)}\Phi_\mp(k) = 2\mathrm{i}k\Psi_{\pm,1}(k), \tag{2.46}$$

where $\Psi_{\pm,1} \in \mathcal{A}$. Finally, $W_\pm(k)/k \in \mathcal{A}$ which together with (2.44) implies that $R_\pm(k) \in \mathcal{A}$.

In the case (b) Lemma 2.6 ensures $K_\pm(x) \in L^1(\mathbb{R}_\pm)$ and, therefore $h_\pm(k) = 2\mathrm{i}k\hat{K}_\pm(k)$, where $\hat{K}_\pm(0) \neq 0$ and $\hat{K}_\pm \in \mathcal{A}$. From (2.27) and (2.28) we get that

$$g_1(k) = h_+(k)h_-(k) + \hat{K}_-(k)h_+'(k) - \hat{K}_+(k)h_-'(k) \in \mathcal{A}_1 \setminus \mathcal{A}$$

and does not have zeros on $\mathbb{R}$ by (2.18). Hence again, $T \in \mathcal{A}_1$. Similarly we obtain $R_\pm \in \mathcal{A}$. □

The next essential lemma is a small variant of the van der Corput lemma(c.f. [12, Cor. 1.1, Page 15]). It can be found in [4, Lemma 5.4]:

Lemma 2.7. *Consider the oscillatory integral*

$$I(t) = \int_a^b \mathrm{e}^{\mathrm{i}t\phi(k)} f(k)dk,$$

where $\phi(k)$ is real-valued function. If $\phi''(k) \neq 0$ in [a,b] and $f \in \mathcal{A}_1$, then

$$|I(t)| \leq C_2[t \min_{a\leq k\leq b} |\phi''(k)|]^{-1/2} \|f\|_{\mathcal{A}_1}, \quad t \geq 1.$$

where $C_2 \leq 2^{8/3}$ is the optimal constant from the van der Corput lemma.

We conclude this section by giving an explicit formula for the kernel of our operator $\mathrm{e}^{-\mathrm{i}tH}P_{ac}(H)$ and naming some results shown in [4], which to a high extend rely on the results of Theorem 2.5. To do so, the next property is crucial, because it gives us an explicit formula for the kernel of the resolvent of H.

Theorem 2.8. *For the kernel of the resolvent $R(\omega)$ of H for $\omega = k^2 \pm \mathrm{i}0$, $k > 0$, the following expression is valid:*

$$[R(k^2 \pm \mathrm{i}0)](x,y) = -\frac{f_+(y,\pm k) f_-(x,\pm k)}{W(\pm k)} = \mp \frac{f_+(y,\pm k) f_-(x,\pm k) T(\pm k)}{2\mathrm{i}k} \tag{2.47}$$

for all $x \le y$ (and the positions of x, y reversed if $x > y$).

We refer , e.g., to [17, Lemma 9.7.]. Next we use Stone's Formula 1.8 to arrive at the following representation:

$$\mathrm{e}^{-\mathrm{i}tH} P_{ac} = \frac{1}{2\pi \mathrm{i}} \int_0^\infty \mathrm{e}^{-\mathrm{i}tk} (R(k + \mathrm{i}0) - R(k - \mathrm{i}0)) dk, \tag{2.48}$$

where we integrate with respect to the spectral measure of H(cf. [17, Sect. 9.3 - 9.7.] for further details on the computation of this measure) and the expression is understood as an improper integral. Next we make the change of variables $k \mapsto k^2$ in (2.48). To do so inside the resolvent, we remark that $R(k + \mathrm{i}0)$ is an analytic continuation of $(H - z)^{-1}$ from the upper half-plane. The continuation of $(H - z^2)^{-1}$ is therefore $(H - (k + \mathrm{i}0)^2)^{-1}$, which is equivalent to $R(k^2 + \mathrm{i}0)$ along the positive and to $R(k^2 - \mathrm{i}0)$ along the negative half line. Therefore it's possible to integrate along the whole real line and in the case $x \le y$ we arrive at the following expression for the kernel of $\mathrm{e}^{-\mathrm{i}tH} P_c$

$$\begin{aligned} [\mathrm{e}^{-\mathrm{i}tH} P_{ac}](x,y) &= \frac{\mathrm{i}}{\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tk^2} \, \frac{f_+(y,k) f_-(x,k) T(k)}{2\mathrm{i}k} k \, dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(tk^2 - |y-x|k)} h_+(y,k) h_-(x,k) T(k) dk. \end{aligned} \tag{2.49}$$

With the results mentioned so far, Egorova, Kopylova, Marchenko and Teschl proved the following theorems on dispersive estimates for Schrödinger operators and improved previous results:

Theorem 2.9. [4, Theorem 1.1] *Let $V \in L^1_1(\mathbb{R})$. Then the following decay holds*

$$\|\mathrm{e}^{-\mathrm{i}tH} P_{ac}\|_{L^1 \to L^\infty} = \mathcal{O}(t^{-1/2}), \quad t \to \infty. \tag{2.50}$$

Moreover, if there is no resonance at $k = 0$, the following holds:

Theorem 2.10. [4, Theorem 1.2] *Let $V \in L^1_2(\mathbb{R})$. Then, in the non-resonant case, the following decay holds*

$$\|\mathrm{e}^{-\mathrm{i}tH} P_{ac}\|_{L^1_1 \to L^\infty_{-1}} = \mathcal{O}(t^{-3/2}), \quad t \to \infty.$$

It should be remarked that if we want to prove e.g. (2.50), we actually need to verify that

$$\left\| \mathrm{e}^{-\mathrm{i}tH} P_{ac} \psi \right\|_\infty \le const \cdot t^{-1/2} \left\| \psi \right\|_1, \quad \psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$$

holds, since we can extend such an estimate to all L^1-functions by an approximation argument. In the next section, we show that a $t^{-3/2}$ time decay is also possible in the resonant case, using stronger assumptions on the potential V.

3. The Resonant Case: Continuous Version

The aim of this section is to prove the following theorem:

Theorem 3.1. *Suppose $V \in L^1_3(\mathbb{R})$ and let f_0 be a bounded function for which $Hf_0 = 0$ and $\lim_{x\to\infty}\left(|f_0(x)|^2 + |f_0(-x)|^2\right) = 2$ holds. Denote by P_0 the projection on the span of f_0 given formally by $P_0\psi = \langle\psi, f_0\rangle f_0$. Then the following decay holds*

$$\|\mathrm{e}^{-\mathrm{i}tH}P_{ac} - (4\pi\mathrm{i}t)^{-\frac{1}{2}}P_0\|_{L^1_2\to L^\infty_{-2}} = \mathcal{O}(t^{-3/2}), \quad t\to\infty. \tag{3.1}$$

To establish this result we need some preparatory lemmas. First of all we show the following:

Lemma 3.2. *If $V \in L^1_2$, we have that $\frac{\partial}{\partial k}h_\pm(x,k)$ and $\frac{\partial}{\partial k}h'_\pm(x,k)$ are contained in $\mathcal{A}$, and for $\pm x \geq 0$ the $\mathcal{A}$-norms of these expressions do not depend on x.*

Proof. Taking the derivative with respect to k in (2.22), we obtain

$$\frac{\partial}{\partial k}h_\pm(x,k) = 2\mathrm{i}\int_0^{\pm\infty} yB_\pm(x,y)\mathrm{e}^{\pm 2\mathrm{i}ky}dy$$

and for $\mp x \leq 0$, $yB_\pm(x,y)$ is integrable with respect to y by (2.23), with L^1-norm not depending on x.
For $h'_\pm(x,k)$ the calculations are exactly the same, we just have to substitute $B_\pm(x,y)$ by $B'_\pm(x,y)$ and $h_\pm(x,k)$ by $h'_\pm(x,k)$ and use (2.24) instead. □

The next result which we need is obtained in a similar way:

Lemma 3.3. *Let $V \in L^1_3$. Then $\frac{\partial}{\partial k}(\frac{h_\pm(x,k)-h_\pm(x,0)}{k})$ and $\frac{\partial}{\partial k}(\frac{h'_\pm(x,k)-h'_\pm(x,0)}{k})$ are contained in $\mathcal{A}$, and for $\pm x \geq 0$ the $\mathcal{A}$-norms of these expressions do not depend on x.*

Proof. We have

$$\begin{aligned} g_{\pm,2}(x,k) = \frac{h_\pm(x,k)-h_\pm(x,0)}{k} &= \pm\int_0^{\pm\infty} B_\pm(x,y)\frac{\mathrm{e}^{\pm 2\mathrm{i}ky}-1}{k}dy \\ &= \pm 2\mathrm{i}\int_0^{\pm\infty} B_\pm(x,z)\left(\int_0^z \mathrm{e}^{\pm 2\mathrm{i}ky}dy\right)dz \end{aligned} \tag{3.2}$$

and denote

$$K_\pm(x,y) = \pm\int_y^{\pm\infty} B_\pm(x,z)dz.$$

Then (3.2) can be written as

$$\pm 2\mathrm{i}\int_0^{\pm\infty}\mathrm{e}^{\pm 2\mathrm{i}ky}\left(\int_y^{\pm\infty}B_\pm(x,z)dz\right)dy = 2\mathrm{i}\int_0^{\pm\infty}K_\pm(x,y)\mathrm{e}^{\pm 2\mathrm{i}ky}dy.$$

Now differentiating with respect to k gives us

$$\frac{\partial}{\partial k}g_{\pm,2}(x,k) = \mp 4\int_0^{\pm\infty} yK_\pm(x,y)\mathrm{e}^{\pm 2\mathrm{i}ky}dy,$$

and for $\mp x \leq 0$, $yK_\pm(x,y)$ is again integrable with respect to y by (2.23), with L^1-norm not depending on x.
For the second task we can proceed exactly in the same way as pointed out at the end of the previous lemma. □

In a similar way we can obtain the next important result:

Lemma 3.4. *If $V \in L^1_3$, we get that $\frac{\partial}{\partial k}(\frac{\Psi_\pm(k)-\Psi_\pm(0)}{k})$ and $\dot{\Psi}_\pm(k)$ lie in $\mathcal{A}$, where $\Psi_\pm(k)$ is given by (2.32).*

Proof. By (2.32) $\Psi_\pm(k)$ can be written as $\int_0^{\pm\infty} H_\pm(y)\mathrm{e}^{\pm 2\mathrm{i}ky}dy$. So we can proceed as in Lemma 3.2 or Lemma 3.3 respectively by using the estimate for $H_\pm$ established in Lemma 2.6. □

A key-tool to prove Theorem 3.1 is the next statement. Therefore we need all the results established so far in this section and we use similar arguments as in Theorem 2.5:

Theorem 3.5. *If $V \in L^1_3$, then $\dot{T}(k)$, $\frac{\partial}{\partial k}(\frac{T(k)-T(0)}{k})$, $\dot{R}_\pm(k)$ and $\frac{\partial}{\partial k}(\frac{R_\pm(k)-R_\pm(0)}{k})$ are elements of $\mathcal{A}$.*

Proof. Throughout the proof of this theorem we use the same notation as in Theorem 2.5. Again we distinguish the two cases (a) $h_+(0)h_-(0) \neq 0$ and (b) $h_+(0) = h_-(0) = 0$.
In case (a) we obtained $W(k) = 2\mathrm{i}k(h_-(k)h_+(k) + \Psi(k))$, where $\Psi(k)$ is given by (2.42). We have $g_1(k) = \frac{1}{T(k)} = \frac{W(k)}{2\mathrm{i}k} = h_-(k)h_+(k) + \Psi(k)$. We then get

$$\frac{T(k)-T(0)}{k} = -\frac{g_1(k)-g_1(0)}{k} \cdot \frac{1}{g_1(k)g_1(0)}.$$

The derivative of this expression can be written as

$$\frac{\partial}{\partial k}\left(-\frac{g_1(k)-g_1(0)}{k}\right) \cdot \frac{1}{g_1(k)g_1(0)} + \frac{g_1(k)-g_1(0)}{k} \cdot \dot{g}_1(k)g_1(0) \cdot \left(\frac{1}{g_1(k)}\right)^2. \quad (3.3)$$

We already know that $\frac{1}{g_1(k)} \in \mathcal{A}_1$. So in order to show that (3.3) lies in the Wiener algebra $\mathcal{A}$, we have to show that $\dot{g}_1(k)$ and $\frac{\partial}{\partial k}\left(\frac{g_1(k)-g_1(0)}{k}\right)$ do.
To establish the first claim a direct calculation shows, that $\dot{g}_1(k)$ is given by

$$\dot{h}_-(k)h_+(k) + h_-(k)\dot{h}_+(k) + \frac{\dot{h}_-(k)}{h_+(0)}\Psi_+(k) + \frac{h_-(k)}{h_+(0)}\dot{\Psi}_+(k) - \frac{\dot{h}_+(k)}{h_-(0)}\Psi_-(k) - \frac{h_+(k)}{h_-(0)}\dot{\Psi}_-(k),$$

and using Lemma 3.2 and Lemma 3.4, we conclude that this expression has to lie in $\mathcal{A}$. For the second claim we rewrite $\frac{g_1(k)-g_1(0)}{k}$ as

$$\frac{h_+(k)-h_+(0)}{k}h_-(k) + \frac{h_-(k)-h_-(0)}{k}h_+(0) + \frac{\Psi_+(k)-\Psi_+(0)}{k}\frac{h_-(k)}{h_+(0)} + \quad (3.4)$$
$$\frac{h_-(k)-h_-(0)}{k}\frac{\Psi_+(0)}{h_+(0)} - \frac{\Psi_-(k)-\Psi_-(0)}{k}\frac{h_+(k)}{h_-(0)} - \frac{h_+(k)-h_+(0)}{k}\frac{\Psi_-(0)}{h_-(0)}.$$

To see that the derivative lies in $\mathcal{A}$ we have to treat each summand separately and use Lemma 3.2, Lemma 3.3 and Lemma 3.4. We only show it for the first one, for the other ones the procedure is exactly the same:

$$\frac{\partial}{\partial k}\left(\frac{h_+(k)-h_+(0)}{k}h_-(k)\right) = \frac{\partial}{\partial k}\left(\frac{h_+(k)-h_+(0)}{k}\right)h_-(k) + \frac{h_+(k)-h_+(0)}{k}\dot{h}_-(k).$$

This expression obviously is contained in $\mathcal{A}$.
To finish the first case it now remains to show that $\frac{\partial}{\partial k}(\frac{R_\pm(k)-R_\pm(0)}{k}) \in \mathcal{A}$. A similar

calculation as in (3.4) shows that $\frac{\partial}{\partial k}(\frac{\Psi_{\pm,1}(k)-\Psi_{\pm,1}(0)}{k}) \in \mathcal{A}$ where $\Psi_{\pm,1}$ is given by (2.46). Since $R_\pm(k) = \mp\Psi_{\pm,1}(k)\cdot T(k)$ by (2.17), we obtain

$$\frac{R_\pm(k)-R_\pm(0)}{k} = \mp\left[\frac{\Psi_{\pm,1}(k)-\Psi_{\pm,1}(0)}{k}T(k) + \frac{T(k)-T(0)}{k}\Psi_{\pm,1}(0)\right].$$

If we take derivatives as before, we are done.
In the case (b) we have $\Psi_\pm(k) = -\hat{K}_\pm(k)h'_\pm(0)$ by (2.33). Since we can assume $h'_\pm(0) \neq 0$, Lemma 3.4 also works for $\hat{K}_\pm(k)$ in this case. Furthermore we have $h_\pm(k) = 2\mathrm{i}k\hat{K}_\pm(k)$ with $\hat{K}_\pm(0) \neq 0$ by Lemma 2.6. $g_1(k)$ is therefore given by $h_+(k)h_-(k) + \hat{K}_-(k)h'_+(k) - \hat{K}_+(k)h'_-(k)$, lies in $\mathcal{A}_1 \setminus \mathcal{A}$ and does not have zeroes on $\mathbb{R}$. This implies $\frac{1}{g_1(k)} \in \mathcal{A}_1$. Taking the derivative with respect to k and using Lemma 3.2 and Lemma 3.4 it immediately follows, that $\dot{g}_1(k)$ lies in $\mathcal{A}$. If we do similar calculations as in (3.4), we arrive at the following expression for $\frac{g_1(k)-g_1(0)}{k}$:

$$\frac{h_+(k)-h_+(0)}{k}h_-(k) + \frac{h_-(k)-h_-(0)}{k}h_+(0) + \frac{\hat{K}_+(k)-\hat{K}_+(0)}{k}h'_-(k) +$$
$$\frac{h'_-(k)-h'_-(0)}{k}\hat{K}_+(0) - \frac{\hat{K}_-(k)-\hat{K}_-(0)}{k}h'_+(k) - \frac{h'_+(k)-h'_+(0)}{k}\hat{K}_-(0).$$

Now $\frac{\partial}{\partial k}(\frac{g_1(k)-g_1(0)}{k}) \in \mathcal{A}$ follows like before, invoking Lemma 3.2, Lemma 3.3 and Lemma 3.4. Following (3.3) we are done.
For $R_\pm(k)$ the calculations are similar. We get that $\Psi_{\pm,1} = \hat{K}_\mp(k)h'_\pm(-k) + K_\pm(-k)h'_\mp(k)$ and therefore $\frac{\partial}{\partial k}(\frac{\Psi_{\pm,1}(k)-\Psi_{\pm,1}(0)}{k}) \in \mathcal{A}$. The rest follows like in case (a). □

Now we continue with the proof of our main Theorem 3.1. To do so, we now give another representation of our projection operator $(4\pi\mathrm{i}t)^{-\frac{1}{2}}P_0$. This result is taken from [6]:

Lemma 3.6. *The Integral kernel of* $(4\pi\mathrm{i}t)^{-\frac{1}{2}}P_0$*, which is (per definition) given by* $(4\pi\mathrm{i}t)^{-\frac{1}{2}}f_0(x)f_0(y)$*, can also be written in the form*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tk^2}T(0)f_-(x,0)f_+(y,0)dk.$$

Proof. It is clear that $f_-(x,0)$ and $f_+(x,0)$ are both scalar multiplies of $f_0(x)$. So we are interested in finding the values of these scalars. First of all we convince ourselves, that in the resonant case $T(0)$ and $R_\pm(0)$ are real-valued: Since by (2.42)–(2.43) $T(k) = h_-(k)h_+(k) + \Psi(k)$ the claim for T follows, if we set $k = 0$ in (2.22) and (2.32). For $R_\pm$ we do the same in (2.46). Next we have a closer look at the limiting values of our Jost solutions $f_\pm(x,0)$. Clearly $f_\pm(x,0)$ goes to 1 as x approaches $\pm\infty$. For the limit $x \to \mp\infty$ we use the scattering relations (2.16). Since all the appearing functions are continuous at $k = 0$, this implies that $f_\pm(x,0)$ converges to $\frac{R_\mp(0)+1}{T(0)}$ in this case.
So if $f_\pm(x,0) = c_\pm f_0(x)$, we get

$$\lim_{x\to\infty}\left(c_\pm^2\,|f_0(x)|^2 + c_\pm^2\,|f_0(-x)|^2\right) = 2c_\pm^2 =$$
$$\lim_{x\to\infty}\left(|f_\pm(x,0)|^2 + |f_\pm(-x,0)|^2\right) = \left(1 + \frac{R_\mp(0)+1}{T(0)}\right)^2$$

and therefore $c_\pm = \sqrt{\frac{1}{2}\cdot\left[\left(\frac{R_\mp(0)+1}{T(0)}\right)^2+1\right]}$. Since $\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}tk^2}dk = (4\pi\mathrm{i}t)^{-\frac{1}{2}}$ the only thing to show, in order to finish the proof, is $c_+c_-T(0) = 1$. To this end we observe that (2.19) gives us $T(0)^2 + R_\pm(0)^2 = 1$ and $R_+(0) = -R_-(0)$. Using these properties we get

$$c_+c_-T(0) = \frac{1}{2T(0)}\sqrt{\left[(R_-(0)+1)^2+T(0)^2\right]\left[(R_+(0)+1)^2+T(0)^2\right]} =$$
$$\frac{1}{2T(0)}\sqrt{[R_-(0)^2+2R_-(0)+1+T(0)^2]\,[R_+(0)^2+2R_+(0)+1+T(0)^2]} =$$
$$\frac{1}{2T(0)}\sqrt{2\,[1+R_-(0)]\,2\,[1-R_-(0)]} = 1.$$

□

Moreover, we need to calculate the following Fourier transform:

Lemma 3.7. *The function* $\frac{\mathrm{e}^{\mathrm{i}kx}-1}{k}$ *and its derivative are contained in* $\mathcal{A}$*, with* $\mathcal{A}$*-norm at most proportional to* $|x|$ *and* $|x|^2$ *respectively.*

Proof. First of all we can easily calculate:

$$\frac{\mathrm{e}^{\mathrm{i}kx}-1}{k} = 2\mathrm{i}\mathrm{e}^{\mathrm{i}k\frac{x}{2}}\frac{\sin(k\frac{x}{2})}{k}.$$

Let $\pm x \geq 0$. Then we get

$$(\frac{\mathrm{e}^{\mathrm{i}kx}-1}{k})\widehat{\;}(p) = \mathrm{i}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kp}\mathrm{e}^{\mathrm{i}k\frac{x}{2}}\frac{\sin(k\frac{x}{2})}{k}dk = \pm\mathrm{i}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}k(\frac{2p}{x}+1)}\frac{\sin k}{k}dk$$
$$= \pm 2\mathrm{i}(\frac{\sin k}{k})\widehat{\;}(\frac{2p}{x}+1),$$

where we used the transformation $k \to k\frac{x}{2}$ to obtain the second equality. The Fourier transform of the characteristic function $\chi_{[-1,1]}$ of the interval $[-1,1]$ can be computed as $(\chi_{[-1,1]}(k))\widehat{\;}(p) = \sqrt{\frac{2}{\pi}}\frac{\sin p}{p}$ by a straightforward calculation. Since $\chi_{[-1,1]}$ is an even function, this allows us to conclude $(\frac{\sin k}{k})\widehat{\;}(p) = \sqrt{\frac{\pi}{2}}\chi_{[-1,1]}(p)$. Therefore $(\frac{\mathrm{e}^{\mathrm{i}kx}-1}{k})\widehat{\;}(p)$ is given by

$$\pm\mathrm{i}\sqrt{2\pi}\chi_{[-1,1]}(\frac{2p}{x}+1) = \pm\mathrm{i}\sqrt{2\pi}\chi_{[-x,0]}(p).$$

Thus the $\mathcal{A}$-norm of $\frac{\mathrm{e}^{\mathrm{i}kx}-1}{k}$ is proportional to $|x|$ and also the claim for the derivative now follows immediately. □

Finally we have all the ingredients needed to obtain Theorem 3.1:

Proof of Theorem 3.1. For the kernel of $\mathrm{e}^{-\mathrm{i}tH}P_{ac}(H)$ we use (2.49):

$$[\mathrm{e}^{-\mathrm{i}tH}P_{ac}](x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}tk^2}\mathrm{e}^{\mathrm{i}|y-x|k}h_+(y,k)h_-(x,k)T(k)dk.$$

By Lemma 3.6 the kernel of $\mathrm{e}^{-\mathrm{i}tH}P_{ac} - (4\pi\mathrm{i}t)^{-\frac{1}{2}}P_0$ can now be written as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}tk^2}(\mathrm{e}^{\mathrm{i}|y-x|k}h_+(y,k)h_-(x,k)T(k) - h_+(y,0)h_-(x,0)T(0))dk. \qquad (3.5)$$

Let

$$g_3(x,y,k) = \mathrm{e}^{\mathrm{i}|y-x|k} h_+(y,k)h_-(x,k)T(k). \tag{3.6}$$

Integrating (3.5) by parts, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tk^2}(g_3(x,y,k)-g_3(x,y,0))dk = \frac{1}{4\pi \mathrm{i}t}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tk^2}\frac{\partial}{\partial k}\left(\frac{g_3(x,y,k)-g_3(x,y,0)}{k}\right)dk. \tag{3.7}$$

Now we apply Lemma 2.7 to get the desired $t^{-\frac{3}{2}}$ time-decay. So in order to finish our proof, it remains to bound the $\mathcal{A}$-norm of $\frac{\partial}{\partial k}\left(\frac{g_3(x,y,k)-g_3(x,y,0)}{k}\right)$ appropriately, which is done in the next lemma: □

Lemma 3.8. *Assume $V \in L^1_3$. Then the $\mathcal{A}$-norm of $\frac{\partial}{\partial k}\left(\frac{g_3(x,y,k)-g_3(x,y,0)}{k}\right)$ is bounded by $C(|x|+|y|)^2$.*

Proof. We assume $x \le y$ and distinguish the cases (i) $x \le 0 \le y$, (ii) $0 \le x \le y$ and (iii) $x \le y \le 0$.
First of all we introduce the function $g_4(x,y,k) = T(k)h_+(y,k)h_-(x,k)$. Then for $\frac{\partial}{\partial k}(\frac{g_3(x,y,k)-g_3(x,y,0)}{k})$ the following representation is valid:

$$\frac{\partial}{\partial k}\left(\frac{\mathrm{e}^{\mathrm{i}(y-x)k}-1}{k}\right)g_4(x,y,k) + \frac{\mathrm{e}^{\mathrm{i}(y-x)k}-1}{k}g_4(x,y,k) + \frac{\partial}{\partial k}\left(\frac{g_4(x,y,k)-g_4(x,y,0)}{k}\right).$$

By Lemma 3.7, the $\mathcal{A}$-norm of $\frac{\mathrm{e}^{\mathrm{i}(y-x)k}-1}{k}$ is bounded by $C(|x|+|y|)$ and that of its derivative by $C(|x|+|y|)^2$. So it remains to consider the $\mathcal{A}$-norm-bounds of $g_4(x,y,k)$, $\frac{\partial}{\partial k}(g_4(x,y,k))$ and $\frac{\partial}{\partial k}\left(\frac{g_4(x,y,k)-g_4(x,y,0)}{k}\right)$.
We start with case (i). $g_4(x,y,k)$ lies in $\mathcal{A}$ with $\mathcal{A}$-norm-bound independent of x and y. The same property holds for $\frac{\partial}{\partial k}(g_4(x,y,k))$, because of Lemma 3.2 and Lemma 3.5 after applying the product rule, and it's also true for the k-derivative of $\frac{g_4(x,y,k)-g_4(x,y,0)}{k}$, since this expression is equivalent to

$$\frac{T(k)-T(0)}{k}h_+(y,k)h_-(x,k) + \frac{h_+(y,k)-h_+(y,0)}{k}h_-(x,k)T(0) + \frac{h_-(x,k)-h_-(x,0)}{k}h_+(y,0)T(0).$$

After taking derivatives and again invoking Lemma 3.2, Lemma 3.3 and Lemma 3.5, we are done in this case. In the cases (ii) and (iii) we use the scattering relations (2.16) to see that the following representations are valid:

$$g_4(x,y,k) = \begin{cases} h_+(y,k)\left(R_+(k)h_+(x,k)\mathrm{e}^{2\mathrm{i}xk} + h_+(x,-k)\right) & 0 \le x \le y, \\ h_-(x,k)\left(R_-(k)h_-(y,k)\mathrm{e}^{-2\mathrm{i}yk} + h_-(y,-k)\right) & x \le y \le 0. \end{cases}$$

$g_4(x,y,k)$ has an $\mathcal{A}$-norm-bound independent of x and y, since for any function $g(k) \in \mathcal{A}$ and any real s we have $g(k)\mathrm{e}^{\mathrm{i}ks} \in \mathcal{A}$ with the norm independent of s.
If we take the derivative with respect to k, again everything is contained in $\mathcal{A}$ by Lemma 3.2 and Lemma 3.5, however we get an addend where the derivatives of $\mathrm{e}^{2\mathrm{i}xk}$ or $\mathrm{e}^{2\mathrm{i}yk}$ occur, so it follows that the $\mathcal{A}$-norm-bound of $\frac{\partial}{\partial k}(g_4(x,y,k))$is at most proportional to $|x|$ or $|y|$ respectively.
Now finally let's have a look at $\frac{\partial}{\partial k}\left(\frac{g_4(x,y,k)-g_4(x,y,0)}{k}\right)$. Here we use the following

equivalent expression for $\frac{g_4(x,y,k)-g_4(x,y,0)}{k}$, (we only consider the case $0 \le x \le y$, for the other one the calculation is similar):

$$\frac{R_+(k)-R_+(0)}{k}h_+(x,k)h_+(y,k)\mathrm{e}^{2\mathrm{i}xk} + \frac{\mathrm{e}^{2\mathrm{i}xk}-1}{k}R_+(0)h_+(x,k)h_+(y,k)+$$
$$\frac{h_+(x,k)-h_+(x,0)}{k}h_+(y,k)R_+(0) + \frac{h_+(y,k)-h_+(y,0)}{k}h_+(x,0)R_+(0)+$$
$$\frac{h_+(y,k)-h_+(y,0)}{k}h_+(x,-k) + \frac{h_+(x,-k)-h_+(x,0)}{k}h_+(y,0).$$

Here again everything is fine after taking derivatives, which means that every addend is an element of $\mathcal{A}$ by Lemma 3.2, Lemma 3.3 and Lemma 3.5. Since the derivative of $\frac{\mathrm{e}^{2\mathrm{i}xk}-1}{k}$ also occurs, the $\mathcal{A}$-norm of $\frac{\partial}{\partial k}(\frac{g_4(x,y,k)-g_4(x,y,0)}{k})$ is at most proportional to $|x|^2$. □

4. Scattering Theory in the Discrete Case

In the discrete case we work in a more general setup using the Jacobi operator $\widetilde{H}$, defined by (0.2), since this is more useful in applications. As mentioned in the beginning, we use the assumption that (0.3) lies in $\ell^1_\sigma(\mathbb{Z})$ for some $0 \le \sigma \le 3$. We just briefly mention the discrete analogon of the Schrödinger equation, which is given by

$$\mathrm{i}\dot{u}(t) = (-\Delta_L + q)\, u(t) = Su(t), \quad t \in \mathbb{R} \tag{4.1}$$

with real potential q. By Δ_L we denote the discrete Laplacian defined via

$$(\Delta_L u)(n) = \frac{1}{2}(u(n+1) - 2u(n) + u(n-1)), \quad n \in \mathbb{Z}.$$

Since the operator S is a special case of $\widetilde{H}$, our results can also be applied to (4.1), but now let's turn back to the investigation of $\widetilde{H}$. In [16, Theorem 1.5] it is proved, that $\widetilde{H}$ is a bounded self-adjoint operator from $\ell^2(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$ and under our assumptions, the absolutely continuous spectrum of $\widetilde{H}$ is given by $[-1,1]$ and the number of eigenvalues is finite (c.f. [16, Theorem 10.4]). Our next step is to investigate the equation

$$\widetilde{H}\psi = \lambda\psi, \quad \lambda \in \mathbb{C}. \tag{4.2}$$

Sometimes it's more convenient to consider this problem on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and its boundary $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ instead of $\mathbb{C}$. To do so, we use the so called Joukovski transformation. Namely we put

$$\lambda = c(z) = \frac{z + z^{-1}}{2}, \quad \lambda \in \mathbb{C} \setminus [-1,1], \tag{4.3}$$

which is one-to one from the domain $\mathbb{C} \setminus [-1,1]$ onto $\mathbb{D}$. Using this setup one can show the following theorem concerning solutions of (4.2):

Theorem 4.1. *There exist Jost solutions $\varphi_\pm(z,n)$ of $H\varphi_\pm(z,n) = \frac{z+z^{-1}}{2}\varphi_\pm(z,n)$ for $0 < |z| \le 1$, which satisfy $\lim_{n\to\pm\infty} \widetilde{\varphi}_\pm(z,n) = 1$ for $\widetilde{\varphi}_\pm(z,n) = \varphi_\pm(z,n) z^{\mp n}$. Moreover $\widetilde{\varphi}_\pm(z,n)$ is holomorphic on $|z| < 1$ and continuous on $|z| \le 1$.*

The proof of this result can be found, e.g., in [16, Theorem 10.2]. In the discrete case similar scattering relations as in the continuous version hold. Let us therefore denote the usual Wronskian by

$$W(f(z,n), g(z,n)) = a(n-1)(f(z,n-1)g(z,n) - g(z,n-1)f(z,n)). \tag{4.4}$$

Introducing the functions

$$W(z) = W(\varphi_+(z,1), \varphi_-(z,1)),\ W_\pm(z) = W(\varphi_\mp(z,1), \varphi_\pm(z^{-1},1)),\ s(z) = \frac{z - z^{-1}}{2\mathrm{i}}$$

one can show that

$$T(z)\varphi_\pm(z,n) = R_\mp(z)\varphi_\mp(z,n) + \varphi_\mp(z^{-1},n), \quad |z| = 1, z^2 \ne 1 \tag{4.5}$$

is valid, where T and $R_\pm$ again as in (2.17) denote the transmission and reflection coefficients, given by

$$T(z) = \frac{\mathrm{i}s(z)}{W(z)}, \quad R_\pm(z) = \mp\frac{W_\pm(z)}{W(z)}. \tag{4.6}$$

We remark that if $|z| = 1$, we often write $z = \mathrm{e}^{\mathrm{i}\theta}$ for $0 \le \theta \le \pi$ and write $\varphi_\pm(z,n) = \varphi_\pm(\theta,n)$, $T(z) = T(\theta), \cdots$ and so on. Further details on all these computations can again be found in [16, Section 10.2]. Again as in the discrete case, the presence of a resonance at $\hat{z} = \pm 1$ is equivalent to the fact, that the Wronskian $W(\hat{z})$ of the two Jost solutions at this point disappears (c.f. [3, Definition 3.5, Lemma 3.6]). Next we state a similar result as Theorem 2.3, which allows us to represent the Jost solutions via a kernel:

Theorem 4.2. *The Jost solutions $\varphi_\pm(z,j)$ can equivalently be written as*

$$\sum_{\ell=j}^{\pm\infty} K_\pm(j,\ell) z^{\pm\ell}, \quad j \in \mathbb{Z}, \quad |z| \le 1, \tag{4.7}$$

where the transformation operators satisfy

$$|K_\pm(j,\ell)| \le C_\pm(j) \sum_{n=\left[\frac{j+\ell}{2}\right]}^{\pm\infty} \left(\left| a(n) - \frac{1}{2} \right| + |b(n)| \right), \tag{4.8}$$

where $[.]$ denotes the floor function. If furthermore $\pm j \ge \mp 1$ holds, we can replace $C_\pm(j)$ by some universal constant $C_\pm(j) \le C$. As an easy implication, $\widetilde{\varphi}_\pm(z,j)$ satisfies

$$\sum_{\ell=0}^{\pm\infty} \widetilde{K}_\pm(j,\ell) z^{\pm\ell}, \quad j \in \mathbb{Z}, \quad |z| \le 1, \quad \widetilde{K}_\pm(j,\ell) = K_\pm(j, \ell + j) \tag{4.9}$$

with the estimate

$$|\widetilde{K}_\pm(j,\ell)| \le C_\pm(j) \sum_{n=j+\left[\frac{\ell}{2}\right]}^{\pm\infty} \left(\left| a(n) - \frac{1}{2} \right| + |b(n)| \right), \tag{4.10}$$

and again $C_\pm(j)$ can be replaced by some universal constant $C_\pm(j) \le C$, if $\pm j \ge \mp 1$ holds.

For the proof we again refer to [16, Section 10.1]. We remark here, that the notation we use is a little bit different from the one in Teschl's book, e.g., we use different transformation operators for the previous theorem. This turns out to be more convenient for us, since other results, which we mention later on, are also based on this notation. If we introduce the Wiener Algebra

$$\widetilde{\mathcal{A}} = \left\{ f(\theta) = \sum_{m \in \mathbb{Z}} \hat{f}_m \mathrm{e}^{\mathrm{i}m\theta} \;\middle|\; \|\hat{f}\|_{\ell^1} < \infty \right\},$$

i.e. the set of all functions with summable Fourier coefficients, and the norm $\|f\|_{\widetilde{\mathcal{A}}} = \|\hat{f}\|_{\ell^1}$ we can deduce that

$$\varphi_\pm(j,z), \widetilde{\varphi}_\pm(j,z) \in \widetilde{\mathcal{A}} \text{ for } |z| = 1 \tag{4.11}$$

with $\|\widetilde{\varphi}(j,z)\|_{\widetilde{\mathcal{A}}}$ independent of j for $\pm j \ge \mp 1$ by the previous theorem, if we assume, that (0.3) is contained in $\ell_1^1(\mathbb{Z})$. It's also important to mention that an analogous version of the Wiener lemma still holds in the discrete case. Moreover it's very convenient for us, that in analogy to Theorem 2.4 a similar version of the Marchenko equation is valid:

Theorem 4.3. *The kernels of the transformation operators satisfy the Marchenko equations*

$$K_\pm(n,m) + \sum_{\ell=n}^{\pm\infty} K_\pm(n,\ell)F_\pm(\ell+m) = \frac{\delta(n,m)}{K_\pm(n,n)}, \quad \pm m \geq \pm n, \tag{4.12}$$

and for $F_\pm(\ell)$ we have the following estimate:

$$|F_\pm(\ell)| \leq C \sum_{n=\left[\frac{\ell}{2}\right]}^{\pm\infty} \Big(\Big|a(n) - \frac{1}{2}\Big| + |b(n)|\Big). \tag{4.13}$$

For the proof of the equation we refer to [16, Section 10.3]. Now we consider a similar result as Theorem 2.5 for the discrete case. We again explain the proof in all detail and extend some of the considerations from [3] and [5]. A very important tool in the computations is the summation by parts formula, which is more or less the discrete analogon to integration by parts:

$$\sum_{\ell=s}^{\pm\infty} \big(f(\ell) - f(\ell\pm 1)\big)v(\ell) = \sum_{\ell=s}^{\pm\infty} f(\ell)\big(v(\ell) - v(\ell\mp 1)\big) + f(s)v(s\mp 1), \tag{4.14}$$

which is true for all $f(\cdot) \in \ell^1(\mathbb{Z}_\pm)$, $\sup_{\ell\in\mathbb{Z}_\pm} |v(\ell)| < \infty$ or vice versa.

Theorem 4.4. [3, Theorem 4.1] *If* (0.3) *is contained in $\ell^1_1(\mathbb{Z})$, we have that $T(z), R_\pm(z) \in \widetilde{\mathcal{A}}$ for $|z| = 1$ (or equivalently: $T(\theta), R_\pm(\theta) \in \widetilde{\mathcal{A}}$ for $-\pi \leq \theta \leq \pi$).*

Proof. The Wronskian $W(z)$ can only vanish at the boundary points of the continuous spectrum, i.e. at $\hat{z} = \pm 1$. As in the continuous version we only focus on the resonant case $W(1)W(-1) = 0$. As a first step we prove the following lemma:

Lemma 4.5. [5, Lemma 4.1] *If* (0.3) *is contained in $\ell^1_1(\mathbb{Z})$, we have that*

$$\breve{W}_\pm(z) = \varphi_\pm(z,1)\varphi_\pm(\hat{z},0) - \varphi_\pm(z,0)\varphi_\pm(\hat{z},1) \in \widetilde{\mathcal{A}}$$

Proof. Let's introduce the expression

$$\Phi^{(j)}_\pm(s) = \sum_{\ell=s}^{\pm\infty} K_\pm(j,\ell)\hat{z}^\ell \tag{4.15}$$

which makes sense because of Theorem 4.2 and the assumption on (0.3). Moreover, since $\hat{z}^{-1} = \hat{z}$, it follows that

$$\Phi^{(j)}_\pm(j) = \varphi_\pm(\hat{z}, j). \tag{4.16}$$

If we apply summation by parts to (4.7), we get

$$\varphi_\pm(z,j) = \sum_{\ell=j}^{\pm\infty} \Phi^{(j)}_\pm(\ell)\big((\hat{z}z)^{\pm\ell} - (\hat{z}z)^{\pm\ell-1}\big) + \Phi^{(j)}_\pm(j)(\hat{z}z)^{\pm j-1}.$$

If we set

$$\zeta(z) = \frac{z-\hat{z}}{z},$$

we obtain

$$\varphi_\pm(z,0) = \zeta(z) \sum_{\ell=\frac{1\pm1}{2}}^{\pm\infty} \Phi_\pm^{(0)}(\ell)(\hat{z}z)^{\pm\ell} + (z\hat{z})^{\frac{\pm1-1}{2}} \varphi_\pm(\hat{z},0), \tag{4.17}$$

$$\varphi_\pm(z,1) = \zeta(z) \sum_{\ell=\frac{1\pm1}{2}}^{\pm\infty} \Phi_\pm^{(1)}(\ell)(\hat{z}z)^{\pm\ell} + (z\hat{z})^{\frac{\pm1-1}{2}} \varphi_\pm(\hat{z},1) \tag{4.18}$$

after a little calculation. Next we multiply (4.17) by $\varphi_\pm(\hat{z},1)$ and (4.18) by $\varphi_\pm(\hat{z},0)$ and then calculate the difference to arrive at

$$\breve{W}_\pm(z) = \varphi_\pm(z,1)\varphi_\pm(\hat{z},0) - \varphi_\pm(z,0)\varphi_\pm(\hat{z},1) = \zeta(z)\widetilde{\Psi}_\pm(z), \tag{4.19}$$

where

$$\widetilde{\Psi}_\pm(z) = \sum_{\ell=\frac{1\pm1}{2}}^{\pm\infty} h_\pm(\ell)(\hat{z}z)^{\pm\ell}, \tag{4.20}$$

$$h_\pm(\ell) = \Phi_\pm^{(1)}(\ell)\varphi_\pm(\hat{z},0) - \Phi_\pm^{(0)}(\ell)\varphi_\pm(\hat{z},1). \tag{4.21}$$

By Theorem 4.2 it follows that $h_\pm(\cdot) \in \ell^\infty(\mathbb{Z}_\pm)$. Thus it remains to show $h_\pm(\cdot) \in \ell^1(\mathbb{Z}_\pm)$. We only do it for h_- here, for h_+ this has already been done in [5, Lemma 4.1]. For $m \le 0$, the transformation operators appearing in (4.21) satisfy the following equation by Theorem 4.3:

$$K_-(1,m) + \sum_{\ell=1}^{-\infty} K_-(1,\ell)F_-(\ell+m) = 0,$$

$$K_-(0,m) + \sum_{\ell=0}^{-\infty} K_-(0,\ell)F_-(\ell+m) = \frac{\delta(0,m)}{K_-(0,0)}.$$

Next we multiply both expressions by $\hat{z}^{-m}$ and sum from $m = s \le 0$ to $-\infty$ to arrive at

$$\Phi_-^{(1)}(s) + \sum_{m=s}^{-\infty}\sum_{\ell=1}^{-\infty} F_-(\ell+m)\hat{z}^{-\ell-m}\left(\Phi_-^{(1)}(\ell) - \Phi_-^{(1)}(\ell-1)\right) = 0,$$

$$\Phi_-^{(0)}(s) + \sum_{m=s}^{-\infty}\sum_{\ell=0}^{-\infty} F_-(\ell+m)\hat{z}^{-\ell-m}\left(\Phi_-^{(0)}(\ell) - \Phi_-^{(0)}(\ell-1)\right) = \frac{\delta(0,s)}{K_-(0,0)}.$$

Now we set $v(\ell) = F_+(\ell)\hat{z}^{-\ell}$ and use summation by parts as it was mentioned in (4.14), which leads us to

$$\Phi_-^{(1)}(s) + \sum_{m=s}^{-\infty}\Bigg(\sum_{\ell=1}^{-\infty}\left(v(\ell+m) - v(\ell+m+1)\right)\Phi_-^{(1)}(\ell) + \Phi_-^{(1)}(1)v(m+2)\Bigg) = 0, \tag{4.22}$$

$$\Phi_-^{(0)}(s) + \sum_{m=s}^{-\infty}\Bigg(\sum_{\ell=0}^{-\infty}\left(v(\ell+m) - v(\ell+m+1)\right)\Phi_-^{(0)}(\ell) + \Phi_-^{(0)}(0)v(m+1)\Bigg) = \frac{\delta(0,s)}{K_-(0,0)}. \tag{4.23}$$

Now for the first equality we set $d_m = \sum_{\ell=1}^{-\infty} v(\ell+m)\Phi_-^{(1)}(\ell)$, and obtain for any $M < s$, that $\sum_{m=s}^{M}(d_m - d_{m+1}) = d_M - d_{s+1}$, which converges to $-d_{s+1}$ as M approaches $-\infty$. If we treat (4.23) the same way and use (4.16), we obtain

$$\Phi_-^{(1)}(s) + \varphi_-(\hat{z},1)\sum_{m=s}^{-\infty} v(m+1) - \sum_{\ell=0}^{-\infty}\Phi_-^{(1)}(\ell)v(\ell+s+1) = 0, \tag{4.24}$$
$$\Phi_-^{(0)}(s) + \varphi_-(\hat{z},0)\sum_{m=s}^{-\infty} v(m+1) - \sum_{\ell=0}^{-\infty}\Phi_-^{(0)}(\ell)v(\ell+s+1) = \frac{\delta(0,s)}{K_-(0,0)}.$$

Next we distinguish two cases. If $\varphi_-(\hat{z},1)\varphi_+(\hat{z},1) \neq 0$ is valid, we multiply the first equation by $\varphi_+(\hat{z},0)$, the second by $\varphi_+(\hat{z},1)$, subtract the second from the first, and use (4.21) to arrive at

$$h_-(s) - \sum_{\ell=0}^{-\infty} h_-(\ell)v(\ell+s+1) = -\frac{\delta(0,s)}{K_-(0,0)}\varphi_-(\hat{z},1). \tag{4.25}$$

Since $h_-(\cdot) \in \ell^\infty(\mathbb{Z}_-)$, we have

$$|v(\ell)| \leq C \sum_{n=\left[\frac{\ell}{2}\right]}^{+\infty}\left(\left|a(n) - \frac{1}{2}\right| + |b(n)|\right), \tag{4.26}$$

if we take (4.13) into account. However, any bounded solution of (4.25), whose kernel satisfies (4.26) already is contained in $\ell^1(\mathbb{Z}_-)$. We prove this fact in detail immediately in the next lemma, again using the successive approximation method in a similar way as in Lemma 2.6, but before let us consider the case $\varphi_-(\hat{z},1) = \varphi_+(\hat{z},1) = 0$. To this end we observe, that $h_-(s) = \varphi_-(\hat{z},0)\Phi_-^{(1)}(s)$, which leads to

$$h_-(s) - \sum_{\ell=0}^{-\infty} h_-(\ell)v(\ell+s-1) = 0, \tag{4.27}$$

by (4.24). So to finish the proof it remains to show $h_-(\cdot) \in \ell^1(\mathbb{Z}_-)$. Indeed we are again able to obtain a result which is a little bit stronger and which we need later on: □

Lemma 4.6. *For $h_\pm(s)$ given by (4.21) we have*

$$|h_\pm(s)| \leq \hat{C} \sum_{n=\left[\frac{s+1}{2}\right]}^{\pm\infty}\left(\left|a(n) - \frac{1}{2}\right| + |b(n)|\right) = \hat{C}\tilde{\eta}(s) \tag{4.28}$$

for some universal constant $\hat{C} > 0$ and $\pm s \geq 2$.

Proof. We again only consider the - case here. Let $-s \geq 2$. Similar to Lemma 2.6 we rewrite (4.25) as

$$h_-(s) - \sum_{\ell=N}^{-\infty} h_-(\ell)v(s+\ell+1) = -\widetilde{H}(s,N), \text{ where}$$

$$\widetilde{H}(s,N) = -\frac{\delta(0,s)}{K_-(0,0)}\varphi_-(\hat{z},1) + \sum_{\ell=0}^{N} h_-(\ell)v(\ell+s+1) \text{ and}$$

$N \leq 0$ such that $C\sum_{\ell=N}^{-\infty}\widetilde{\eta}(\ell) < 1$ and C given by (4.26). The estimate

$$|\widetilde{H}(s,N)| \leq C(N)\widetilde{\eta}(s), \quad s \leq -2 \tag{4.29}$$

follows from monotonicity of $\widetilde{\eta}$ and $h_-(\cdot) \in \ell^\infty(\mathbb{Z}_-)$. Now we set

$$h_{-,0}(s) = \widetilde{H}(s,N), h_{-,n+1}(s) = \sum_{\ell=N}^{-\infty} h_{-,n}(\ell)v(\ell+s+1).$$

We show that

$$|h_{-,n}(s)| \leq C(N)\widetilde{\eta}(s)\left[C\sum_{\ell=N}^{-\infty}\widetilde{\eta}(\ell)\right]^n, \tag{4.30}$$

where $C(N)$ is given by (4.29). Then we are done using similar methods as in Theorem 2.1, and the desired estimate for $h_-(s)$ also follows immediately, because

$$|h_-(s)| \leq C(N)\widetilde{\eta}(s)\sum_{n=0}^{\infty}\left[C\sum_{\ell=N}^{-\infty}\widetilde{\eta}(\ell)\right]^n = \hat{C}(N)\widetilde{\eta}(s).$$

So it remains to verify (4.30) by induction. (4.29) ensures the estimate for $n = 0$. Now assume it is true for n. Then, using (4.26) and monotonicity of $\widetilde{\eta}(s)$, we finally get

$$\begin{aligned}
|h_{-,n+1}(s)| &\leq C(N)\sum_{\ell=N}^{-\infty}\widetilde{\eta}(\ell)\left[C\sum_{k=N}^{-\infty}\widetilde{\eta}(k)\right]^n |v(\ell+s+1)| \\
&\leq C(N)\left[C\sum_{k=N}^{-\infty}\widetilde{\eta}(k)\right]^n \sum_{\ell=N}^{-\infty}\widetilde{\eta}(\ell)C\widetilde{\eta}(\ell+s) \\
&\leq C(N)\widetilde{\eta}(s)\left[C\sum_{\ell=N}^{-\infty}\widetilde{\eta}(\ell)\right]^{n+1}.
\end{aligned}$$

□

Now we can continue with the proof of Theorem 4.4. Let us first assume we have resonance at only one point $\hat{z}$. We distinguish the cases (a) $\varphi_+(\hat{z},0)\varphi_-(\hat{z},0) \neq 0$ and (b) $\varphi_+(\hat{z},1)\varphi_-(\hat{z},1) \neq 0$, since the solutions $\varphi_\pm(\hat{z},m)$ cannot vanish at two consecutive points. First we consider case (a). Then a little calculation, similar to the one in (2.42), shows

$$\begin{aligned}
W(z) &= \varphi_+(z,0)\varphi_-(z,0)\left(\frac{\breve{W}_-(z)}{\varphi_-(\hat{z},0)\varphi_-(z,0)} - \frac{\breve{W}_+(z)}{\varphi_+(\hat{z},0)\varphi_+(z,0)}\right) = \\
&= \zeta(z)\left(\frac{\varphi_+(z,0)}{\varphi_-(\hat{z},0)}\widetilde{\Psi}_-(z) - \frac{\varphi_-(z,0)}{\varphi_+(\hat{z},0)}\widetilde{\Psi}_+(z)\right) = \zeta(z)\widetilde{\Psi}(z),
\end{aligned} \tag{4.31}$$

where $\widetilde{\Psi}(z) \in \widetilde{\mathcal{A}}$ by Lemma 4.5 and (4.11). Since $|T(z)| \le 1$, the zeroes of $W(z)$ can at most be of first order, which implies $\widetilde{\Psi}(z) \neq 0$ on $[-1,1]$. Thus the Wiener lemma implies, that $T(z) \in \widetilde{\mathcal{A}}$. To get the same result for $R_\pm(z)$, we observe that $W(\hat{z}) = 0$ implies $W_\pm(\hat{z}) = 0$ and thus $W^\pm(z) = \zeta(z)\widetilde{\Psi}_{\pm,1}(z)$ with $\widetilde{\Psi}_{\pm,1}(z) \in \widetilde{\mathcal{A}}$ and $\widetilde{\Psi}_{\pm,1}(z) \neq 0$ on $[-1,1]$, using similar calculations as before. For case (b) the calculations are similar. If there is resonance at both points -1 and 1, by the previous considerations $T(z)$ has two Fourier series expansions with summable Fourier coefficients. These expansions are valid on $\mathbb{T}$ without point -1 or 1 respectively, and they coincide everywhere, but on these two exceptional points. Thus they exist and coincide also on the exceptional points and we are done also in this case. □

We again conclude this section by considering the propagator $\mathrm{e}^{-\mathrm{i}t\widetilde{H}}P_{ac}$. For the resolvent, a similar formula as in the continuous case (c.f. (2.47)) is valid:

Theorem 4.7. *(cf. [16, (1.99)]) The kernel of the resolvent $R(\lambda)$ for $z \in \mathbb{D}$ and λ given by* (4.3)*, can be expressed as*

$$[R(\lambda)](n,k) = \frac{1}{W(z)} \begin{cases} \varphi_+(z,n)\varphi_-(z,k) & \text{for } n \ge k, \\ \varphi_+(z,k)\varphi_-(z,n) & \text{for } n \le k. \end{cases} \tag{4.32}$$

Next we apply Stone's Formula 1.8 again to arrive at the following representation:

$$\mathrm{e}^{-\mathrm{i}t\widetilde{H}}P_{ac} = \frac{1}{2\pi\mathrm{i}}\int_{-1}^{1} \mathrm{e}^{-\mathrm{i}t\lambda}(R(\lambda+\mathrm{i}0) - R(\lambda-\mathrm{i}0))d\lambda, \tag{4.33}$$

where we integrate with respect to the spectral measure of $\widetilde{H}$. Using (4.3) and the fact that this transformation is a conformal map(so, especially, it preserves orientation, and, heuristically, maps boundaries to boundaries), with the help of formula (4.32), the kernel of (4.33) reads

$$\begin{aligned}\left[\mathrm{e}^{-\mathrm{i}t\widetilde{H}}P_{ac}\right](n,k) =& \frac{1}{2\pi\mathrm{i}}\int_{z_+} \mathrm{e}^{-\mathrm{i}tc(z)}\frac{\varphi_+(z,n)\varphi_-(z,k)}{W(z)}\frac{\mathrm{i}s(z)}{z}dz + \\ & \frac{1}{2\pi\mathrm{i}}\int_{z_-} \mathrm{e}^{-\mathrm{i}tc(z)}\frac{\varphi_+(z,n)\varphi_-(z,k)}{W(z)}\frac{\mathrm{i}s(z)}{z}dz,\end{aligned} \tag{4.34}$$

where z_+ denotes the positively oriented curve along the upper half of the unit circle, i.e.

$$\begin{aligned} z_+ : [0,\pi] &\to \mathbb{C} \\ \theta &\mapsto \mathrm{e}^{\mathrm{i}\theta}, \end{aligned}$$

whereas $\mathbb{T}_-$ is the negatively oriented analogon with $\theta \in [-\pi, 0]$. Since in terms of λ, $R(\lambda + \mathrm{i}0)$ is the continuation of the resolvent from the upper half-plane to the interval $[-1,1]$, in terms of z(i.e. under the Joukovski transform) this process can be seen as a continuation of $R(c(z))$ from $\mathbb{D}$ to the boundary $\mathbb{T}_+$. Similarly for $R(\lambda - \mathrm{i}0)$ and $\mathbb{T}_-$. This explains (4.34). If we substitute $z = \mathrm{e}^{\mathrm{i}\theta}$, (4.34) can once more be transformed to

$$\begin{aligned} &\frac{\mathrm{i}}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}t\cos\theta}\frac{\varphi_+(\theta,n)\varphi_-(\theta,k)}{W(\theta)}\sin\theta\, d\theta = \\ &\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}t\cos\theta}\varphi_+(\theta,n)\varphi_-(\theta,k)T(\theta)d\theta, \end{aligned} \tag{4.35}$$

by (4.6). To get the desired time-decay estimates, the van der Corput lemma is essential again. Similar to Lemma 2.7 we need a small variant of this important result, which can be found in [3, Lemma 5.1]:

Lemma 4.8. *Consider the oscillatory integral*

$$I(t) = \int_a^b \mathrm{e}^{\mathrm{i}t\phi(\theta)} f(\theta) d\theta, \qquad -\pi \le a < b \le \pi,$$

where $\phi(\theta)$ *is real-valued. If* $\min_{\theta\in[a,b]} |\phi^{(s)}(\theta)| = m_s > 0$ *for some* $s \ge 2$ *and* $f \in \mathcal{A}$, *then*

$$|I(t)| \le \frac{C_s \|\hat{f}\|_{\ell^1}}{(m_s t)^{1/s}}, \quad t \ge 1,$$

where C_s *is a universal constant.*

Now, finally, these are all the preliminaries we need to establish the desired integrable time decay for $\widetilde{H}$ in the resonant case. To conclude this section we briefly mention some recent results on discrete Schrödinger operators, which are proved in [3]:

Theorem 4.9. *Let* $q \in \ell^1_1$. *Then the asymptotics*

$$\|\mathrm{e}^{-\mathrm{i}tS} P_{ac}\|_{\ell^1 \to \ell^\infty} = \mathcal{O}(t^{-1/3}), \quad t \to \infty \tag{4.36}$$

hold.

Theorem 4.10. *Let* $q \in \ell^1_2$. *Then in the non-resonant case the following asymptotics hold:*

$$\|\mathrm{e}^{-\mathrm{i}tS} P_{ac}\|_{\ell^1_1 \to \ell^\infty_{-1}} = \mathcal{O}(t^{-4/3}), \quad t \to \infty.$$

We mentioned in the beginning of this section, that discrete Schrödinger operators can be treated in a similar way as $\widetilde{H}$. So, in particular, if for $\widetilde{H}$ a $t^{-4/3}$ time decay in the resonant case is valid, it's clear that this result also holds for the Schrödinger operator S. On the other hand, one can also apply the results of the last theorems to $\widetilde{H}$. Finally we remark that an estimate of type (4.36) can be established in the same way as it was mentioned at the end of Section 2.

5. The Resonant Case: Discrete Version

The main result of this section reads as follows:

Theorem 5.1. *Let $\widetilde{H}$ be defined by* (0.2)*, such that* (0.3) *is contained in $\ell^1_3(\mathbb{Z})$. Furthermore assume that there is resonance at the points $\hat{z} = \pm 1$, and there exist bounded solutions $\varphi_{\hat{z}}(n)$ of $\widetilde{H}\varphi_{\hat{z}}(n) = \hat{z}\varphi_{\hat{z}}(n)$ such that*

$$\lim_{n\to+\infty}\left(|\varphi_{\hat{z}}(n)|^2 + |\varphi_{\hat{z}}(-n)|^2\right) = 2$$

holds. Denote by $P_{\hat{z}}$ the projection on the span of $\varphi_{\hat{z}}$ given by the kernel $[P_{\hat{z}}](n,k) = \varphi_{\hat{z}}(n)\varphi_{\hat{z}}(k)$, and $P_{\hat{z}} = 0$, if $\hat{z}$ is no resonance point. Then the following time-decay holds:

$$\|\mathrm{e}^{-\mathrm{i}t\widetilde{H}}P_{ac} - \frac{\mathrm{e}^{-\mathrm{i}t}}{\sqrt{-2\pi\mathrm{i}t}}P_1 - \frac{\mathrm{e}^{\mathrm{i}t}}{\sqrt{2\pi\mathrm{i}t}}P_{-1}\|_{\ell^1_2\to\ell^\infty_{-2}} = \mathcal{O}(t^{-4/3}), \quad t\to\infty. \tag{5.1}$$

Before we prove this theorem, let's start with a little remark. Namely the reason for the more complicated structure of the projectors, compared to the continuous case, is, that when we perform integration by parts to get the desired time decay(c.f. proof of Theorem 3.1), there occurs a boundary term with a factor $\frac{1}{t}$, which finally cancels out due to our choice of the projectors. The procedure in this section is now more or less the same than it was in Section 3. Thus we start with some lemmas, regarding the derivatives of the Jost solutions. As in the previous section, we switch between the representations of the solutions of (4.2) in the variables $z \in \mathbb{T}$ and $\theta \in [-\pi,\pi]$, and use the one, which is most convenient for the calculations, according to the situation.

Lemma 5.2. *If* (0.3) *is contained in ℓ^1_2, we have that $\frac{\partial}{\partial z}(\widetilde{\varphi}_\pm(z,n)) = \widetilde{\varphi}'_\pm(z,n)$ is contained in $\widetilde{\mathcal{A}}$, and for $\pm n \le \mp 1$ the $\widetilde{\mathcal{A}}$-norm of this expression does not depend on n.*

Proof. This follows immediately, if we first differentiate (4.9), then use (4.10) and the assumption on (0.3). □

The next result is the discrete analogon of Lemma 3.3:

Lemma 5.3. *Let* (0.3) *be contained in ℓ^3_1. Let $C \subset \mathbb{T}$ be closed and connected with $-\hat{z} \notin C$. Then, for $z \in C$, it follows that $\frac{\partial}{\partial z}(\frac{\widetilde{\varphi}_\pm(z,n)-\widetilde{\varphi}_\pm(\hat{z},n)}{s(z)}) \in \widetilde{\mathcal{A}}$, and for $\pm n \le \mp 1$, the $\widetilde{\mathcal{A}}$-norm of this expression does not depend on n.*

Proof. Let us denote

$$\widetilde{g}_{\pm,1}(z,n) = \frac{\widetilde{\varphi}_\pm(z,n) - \widetilde{\varphi}_\pm(\hat{z},n)}{s(z)},$$

$$\widetilde{g}_{\pm,2}(z,n) = \frac{\widetilde{\varphi}_\pm(z,n) - \widetilde{\varphi}_\pm(\hat{z},n)}{z-\hat{z}} \quad \text{and}$$

$$\widetilde{g}_3(z) = \frac{2\mathrm{i}z(z-\hat{z})}{z^2-1} = \frac{2\mathrm{i}z}{z+\hat{z}}.$$

By (4.9) we have that

$$\widetilde{g}_{\pm,2}(z,n) = \sum_{\ell=\pm 1}^{\pm\infty} \widetilde{K}_\pm(n,\ell)\frac{z^{\pm\ell}-\hat{z}^{\pm\ell}}{z-\hat{z}} = \sum_{\ell=\pm 1}^{\pm\infty} \widetilde{K}_\pm(n,\ell) \sum_{k=0}^{\pm\ell-1} z^k \hat{z}^{\pm\ell-1-k}$$
$$= \sum_{\ell=0}^{\pm\infty}\Big(\sum_{k=\ell\pm 1}^{\pm\infty} \widetilde{K}_\pm(n,k)\hat{z}^{\ell+k-1}\Big)z^{\pm\ell}.$$

In order to show, that $\widetilde{g}'_{\pm,2}(z,n) \in \widetilde{\mathcal{A}}$, we need to verify that $\ell\sum_{k=\ell\pm1}^{\pm\infty}|\widetilde{K}_\pm(n,k)|$ is summable. This follows, if we invoke (4.10) and use the assumption on (0.3). Furthermore we also get, that the $\widetilde{\mathcal{A}}$-norm is independent from n, if $\pm n \le \mp 1$. So it remains to consider the expression $\widetilde{g}_3(z)$ and its derivative. But on C, this is a smooth function and thus it can be extended to a smooth function on all of $\mathbb{T}$. This especially shows, that $\widetilde{g}_3(z)$ and its derivative are contained in $\widetilde{\mathcal{A}}$ for $z \in C$. Using the product rule on $\widetilde{g}_{\pm,1}(z,n) = \widetilde{g}_{\pm,2}(z,n)\widetilde{g}_3(z)$ finishes the proof. □

In a similar way, we can compute the following result:

Lemma 5.4. *Let* (0.3) *be contained in* ℓ^3_1. *Let* $C \subset \mathbb{T}$ *be closed and connected with* $-\hat{z} \notin C$. *Then, for* $z \in C$, *it follows that* $\frac{\partial}{\partial z}(\frac{\widetilde{\psi}_\pm(z)-\widetilde{\psi}_\pm(\hat{z})}{s(z)})$ *and* $\widetilde{\psi}'_\pm(z)$ *are elements of* $\widetilde{\mathcal{A}}$, *where* $\widetilde{\psi}_\pm(z)$ *are defined in* (4.20).

Proof. By (4.20) we have $\widetilde{\Psi}_\pm(z) = \sum_{\ell=\frac{1\pm1}{2}}^{\pm\infty} h_\pm(\ell)(\hat{z}z)^{\pm\ell}$. So we can use the same arguments as in Lemma 5.2 and Lemma 5.3, because of the estimate on $h_\pm$ established in Lemma 4.6. □

Also the next result now follows in a similar manner as Theorem 3.5:

Theorem 5.5. *Let* (0.3) *be contained in* ℓ^3_1. *Let* $C \subset \mathbb{T}$ *be closed and connected with* $-\hat{z} \notin C$. *Then, for* $z \in C$, *it follows that* $\frac{\partial}{\partial z}(\frac{T(z)-T(\hat{z})}{s(z)})$, $T'(z)$, $\frac{\partial}{\partial z}(\frac{R_\pm(z)-R_\pm(\hat{z})}{s(z)})$ *and* $R'_\pm(z)$ *are elements of* $\widetilde{\mathcal{A}}$.

Proof. We again use the same notation as in Theorem 4.4. First of all let us look at case (a) $\varphi_+(\hat{z},0)\varphi_-(\hat{z},0) \neq 0$. Let's denote $\widetilde{g}_4(z) = \frac{1}{T(z)} = \frac{2\widetilde{\psi}(z)}{z+\hat{z}} = \widetilde{\psi}(z)\widetilde{g}_5(z)$ by (4.31). Then

$$\frac{T(z)-T(\hat{z})}{s(z)} = -\frac{\widetilde{g}_4(z)-\widetilde{g}_4(\hat{z})}{s(z)}\cdot\frac{1}{\widetilde{g}_4(z)\widetilde{g}_4(\hat{z})}. \tag{5.2}$$

We already know that $\frac{1}{\widetilde{g}_4(z)} \in \widetilde{\mathcal{A}}$. So in order to show that the derivative of(5.2) lies in the Wiener algebra $\widetilde{\mathcal{A}}$ for $z \in C$, we have to verify that $\widetilde{g}'_4(z)$ and $\frac{\partial}{\partial z}(\frac{\widetilde{g}_4(z)-\widetilde{g}_4(\hat{z})}{s(z)})$ do.

To establish the first claim we remind that we can say $\widetilde{g}_5(z)$ and all its derivatives have to lie in $\widetilde{\mathcal{A}}$ for every $z \in C$ (c.f. Lemma 5.3). Now having a closer look at (4.31) and using Lemma 5.2 and Lemma 5.4, we conclude that $\widetilde{g}'_4(z) \in \widetilde{\mathcal{A}}$. For the second claim we rewrite $\frac{\widetilde{g}_4(z)-\widetilde{g}_4(\hat{z})}{s(z)}$ as

$$g_{+,1}(z,0)\frac{\widetilde{\psi}_-(z)g_5(z)}{\widetilde{\varphi}_-(\hat{z},0)} + \frac{\widetilde{\psi}_-(z)-\widetilde{\psi}_-(\hat{z})}{s(z)}\frac{\widetilde{\varphi}_+(\hat{z},0)g_5(z)}{\widetilde{\varphi}_-(\hat{z},0)} - g_3(z)\frac{\widetilde{\psi}_-(z)\widetilde{\varphi}_+(\hat{z},0)}{\widetilde{\varphi}_-(\hat{z},0)} -$$
$$g_{-,1}(z,0)\frac{\widetilde{\psi}_+(z)g_5(z)}{\widetilde{\varphi}_+(\hat{z},0)} - \frac{\widetilde{\psi}_+(z)-\widetilde{\psi}_+(\hat{z})}{s(z)}\frac{\widetilde{\varphi}_-(\hat{z},0)g_5(z)}{\widetilde{\varphi}_+(\hat{z},0)} + g_3(z)\frac{\widetilde{\psi}_+(z)\widetilde{\varphi}_+(\hat{z},0)}{\widetilde{\varphi}_+(\hat{z},0)},$$

which gives us all we need if we invoke Lemma 5.2, Lemma 5.3 and Lemma 5.4 and if we use $\varphi_-(z,0) = \widetilde{\varphi}_-(z,0)$. The claim for $R_\pm(z)$ follows in a similar manner as in Theorem 3.5, and in case (b) $\varphi_+(\hat{z},1)\varphi_-(\hat{z},1) \neq 0$ we also use similar computations as before. □

In the next lemma we have a closer look at our projectors:

Lemma 5.6. *Using the asymptotics of the Hankel functions (cf.* [14, 10.17.5, 10.17.6]*), the following expressions for the kernels of our Projectors $P_{\hat{z}}$ are valid:*

$$\frac{\mathrm{e}^{-\mathrm{i}t}}{\sqrt{-2\pi\mathrm{i}t}}[P_1](n,k) = \varphi_+(0,n)\varphi_-(0,k)T(0)\left[\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{e}^{-\mathrm{i}t\cos\theta}d\theta + \frac{1}{\mathrm{i}t\pi}\right] + \mathcal{O}(t^{-\frac{3}{2}}) \tag{5.3}$$

$$\frac{\mathrm{e}^{\mathrm{i}t}}{\sqrt{2\pi\mathrm{i}t}}[P_{-1}](n,k) = \varphi_+(\pi,n)\varphi_-(\pi,k)T(\pi)\left[\frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{e}^{-\mathrm{i}t\cos\theta}d\theta - \frac{1}{\mathrm{i}t\pi}\right] + \mathcal{O}(t^{-\frac{3}{2}}). \tag{5.4}$$

Proof. To show that $[P_{\hat{z}}](n,k) = \varphi_+(\hat{z},n)\varphi_-(\hat{z},k)T(\hat{z})$ the same computations as in Lemma 3.6 work, since continuity for $R_\pm(z), T(z)$ at the resonance points $\hat{z}$ also holds in the discrete case. For the reflection coefficient, this was established in [5, Lemma 4.1]. By [14, 10.9] and [14, 11.5], we get

$$\frac{1}{2}(J_0(t) - \mathrm{i}H_0(t)) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{e}^{-\mathrm{i}t\cos\theta}d\theta,$$

where $J_0(t)$ denotes the Bessel function, and $H_0(t)$ the Struve function, both of order 0(c.f. [14, 10.2] and [14, 11.2]). Moreover, by [14, 11.6] the following asymptotics hold

$$H_0(t) - Y_0(t) = \frac{2}{\pi t} + \mathcal{O}(t^{-3}),$$

where $Y_0(t)$ denotes the Neumann function of order 0 (c.f. [14, 10.2]). Since by [14, 10.4] $H_0^{(1)}(t) = J_0(t) + \mathrm{i}Y_0(t)$ and $H_0^{(2)}(t) = J_0(t) - \mathrm{i}Y_0(t)$, the claim follows using the asymptotics [14, 10.17.5, 10.17.6]. □

We also need an analogon of Lemma 3.7

Lemma 5.7. *Let $C \subset \mathbb{T}$ be closed and connected with $-\hat{z} \notin C$. Then, for $z \in C$, it follows that $\frac{z^n - \hat{z}^n}{s(z)}$ and its derivative are contained in $\widetilde{\mathcal{A}}$, where the $\widetilde{\mathcal{A}}$-norms are at most proportional to $|n|$ or $|n|^2$, respectively.*

Proof. We have that

$$\frac{z^n - \hat{z}^n}{s(z)} = \widetilde{g}_3(z)\frac{z^n - \hat{z}^n}{z - \hat{z}} = \widetilde{g}_3(z)\sum_{\ell=0}^{n-1} \hat{z}^{n-1-\ell}z^\ell.$$

We already know by Lemma 5.3, that we can assume $\widetilde{g}_3(z)$, $\widetilde{g}_3'(z) \in \widetilde{\mathcal{A}}$. The sum in the last expression is also obviously an element of $\widetilde{\mathcal{A}}$ with norm $|n|$. Taking derivatives, we are done, since $\sum_{\ell=0}^{n-1} \ell$ is at most proportional to $|n|^2$. □

Since we have got everything we need, it's time to prove the main theorem of this section now:

Proof of Theorem 5.1. Assume w.l.o.g $n \le k$.
Using (4.35), we can write $[\mathrm{e}^{-\mathrm{i}t\widetilde{H}}P_{ac}](n,k)$ as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}t\cos\theta}\mathrm{e}^{\mathrm{i}|n-k|\theta}\widetilde{\varphi}_+(\theta,n)\widetilde{\varphi}_-(\theta,k)T(\theta)d\theta.$$

Taking advantage of the fact that all the appearing functions are 2π-periodic, we can integrate on the interval $[-\frac{\pi}{2},\frac{3\pi}{2}]$ instead without changing anything. If we denote

$$\widetilde{g}_6(\theta,n,k) = \mathrm{e}^{\mathrm{i}|n-k|\theta}\widetilde{\varphi}_+(\theta,n)\widetilde{\varphi}_-(\theta,k)T(\theta),$$

by Lemma 5.6, the kernel

$$[\mathrm{e}^{-\mathrm{i}t\widetilde{H}}P_{ac} - \frac{\mathrm{e}^{-\mathrm{i}t}}{\sqrt{-2\pi\mathrm{i}t}}P_1 - \frac{\mathrm{e}^{\mathrm{i}t}}{\sqrt{2\pi\mathrm{i}t}}P_{-1}](n,k) \tag{5.5}$$

reads

$$\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{e}^{-\mathrm{i}t\cos\theta}(\widetilde{g}_6(\theta,n,k)-\widetilde{g}_6(0,n,k)d\theta + \widetilde{g}_6(0,n,k)\frac{1}{\mathrm{i}t\pi} \tag{5.6}$$

$$+\frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{e}^{-\mathrm{i}t\cos\theta}(\widetilde{g}_6(\theta,n,k)-\widetilde{g}_6(\pi,n,k)d\theta - \widetilde{g}_6(\pi,n,k)\frac{1}{\mathrm{i}t\pi} + \mathcal{O}(t^{-\frac{3}{2}}) \tag{5.7}$$

If we integrate (5.6) by parts similar to Theorem 3.1, we obtain

$$\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{e}^{-\mathrm{i}t\cos\theta}(\widetilde{g}_6(\theta,n,k)-\widetilde{g}_6(0,n,k)d\theta =$$
$$\frac{1}{2\pi\mathrm{i}t}\left[\mathrm{e}^{-\mathrm{i}t\cos\theta}\cdot\frac{\widetilde{g}_6(\theta,n,k)-\widetilde{g}_6(0,n,k)}{\sin\theta}\right]_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$-\frac{1}{2\pi\mathrm{i}t}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{e}^{-\mathrm{i}t\cos\theta}\frac{\partial}{\partial z}\left(\frac{\widetilde{g}_6(\theta,n,k)-\widetilde{g}_6(0,n,k)}{\sin\theta}\right)d\theta.$$

If we do the same for (5.7) and compute the boundary terms, we get

$$\frac{1}{2\pi\mathrm{i}t}\left[\mathrm{e}^{-\mathrm{i}t\cos\theta}\frac{\widetilde{g}_6(\theta,n,k)-\widetilde{g}_6(0,n,k)}{\sin\theta}\right]_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$=\frac{1}{2\pi\mathrm{i}t}\left(-2\widetilde{g}_6(0,n,k)+\widetilde{g}_6(\frac{\pi}{2},n,k)+\widetilde{g}_6(-\frac{\pi}{2},n,k)\right)$$

and

$$\frac{1}{2\pi\mathrm{i}t}\left[\mathrm{e}^{-\mathrm{i}t\cos\theta}\frac{\widetilde{g}_6(\theta,n,k)-\widetilde{g}_6(\pi,n,k)}{\sin\theta}\right]_{\theta=\frac{\pi}{2}}^{\frac{3\pi}{2}}$$
$$=\frac{1}{2\pi\mathrm{i}t}\left(-\widetilde{g}_6(\frac{3\pi}{2},n,k)-\widetilde{g}_6(\frac{\pi}{2},n,k)+2\widetilde{g}_6(\pi,n,k)\right).$$

Thus all the terms of order $\frac{1}{t}$ vanish and we can rewrite (5.5) as

$$\frac{\mathrm{i}}{2\pi t}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{e}^{-\mathrm{i}t\cos\theta}\frac{\partial}{\partial z}\left(\frac{\widetilde{g}_6(\theta,n,k)-\widetilde{g}_6(0,n,k)}{\sin\theta}\right)d\theta+ \tag{5.8}$$
$$\frac{\mathrm{i}}{2\pi t}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{e}^{-\mathrm{i}t\cos\theta}\frac{\partial}{\partial z}\left(\frac{\widetilde{g}_6(\theta,n,k)-\widetilde{g}_6(\pi,n,k)}{\sin\theta}\right)d\theta + \mathcal{O}(t^{-\frac{3}{2}}),$$

where we neglect the summand of order $t^{-\frac{3}{2}}$ from now on. The other two summands are treated separately and we show the desired $t^{-\frac{4}{3}}$ time decay for each expression. Since the calculations are similar, we only focus on (5.8). Next, to apply Lemma 4.8, we split the domain of integration into parts where either the second or third derivative of the phase $-it\cos\theta$ is nonzero. This gives us the time decay. It remains to show that $\frac{\partial}{\partial z}\left(\frac{\widetilde{g}_6(\theta,n,k)-\widetilde{g}_6(0,n,k)}{\sin\theta}\right) \in \widetilde{\mathcal{A}}$ for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, with $\widetilde{\mathcal{A}}$-norm at most proportional to $(|n|+|k|)^2$. But this can be proved in completely the same way as it was done in Lemma 3.8. The same procedure of course also works for $\frac{\widetilde{g}_6(\theta,n,k)-\widetilde{g}_6(\pi,n,k)}{\sin\theta}$, with $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. $\square$

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